The Variable

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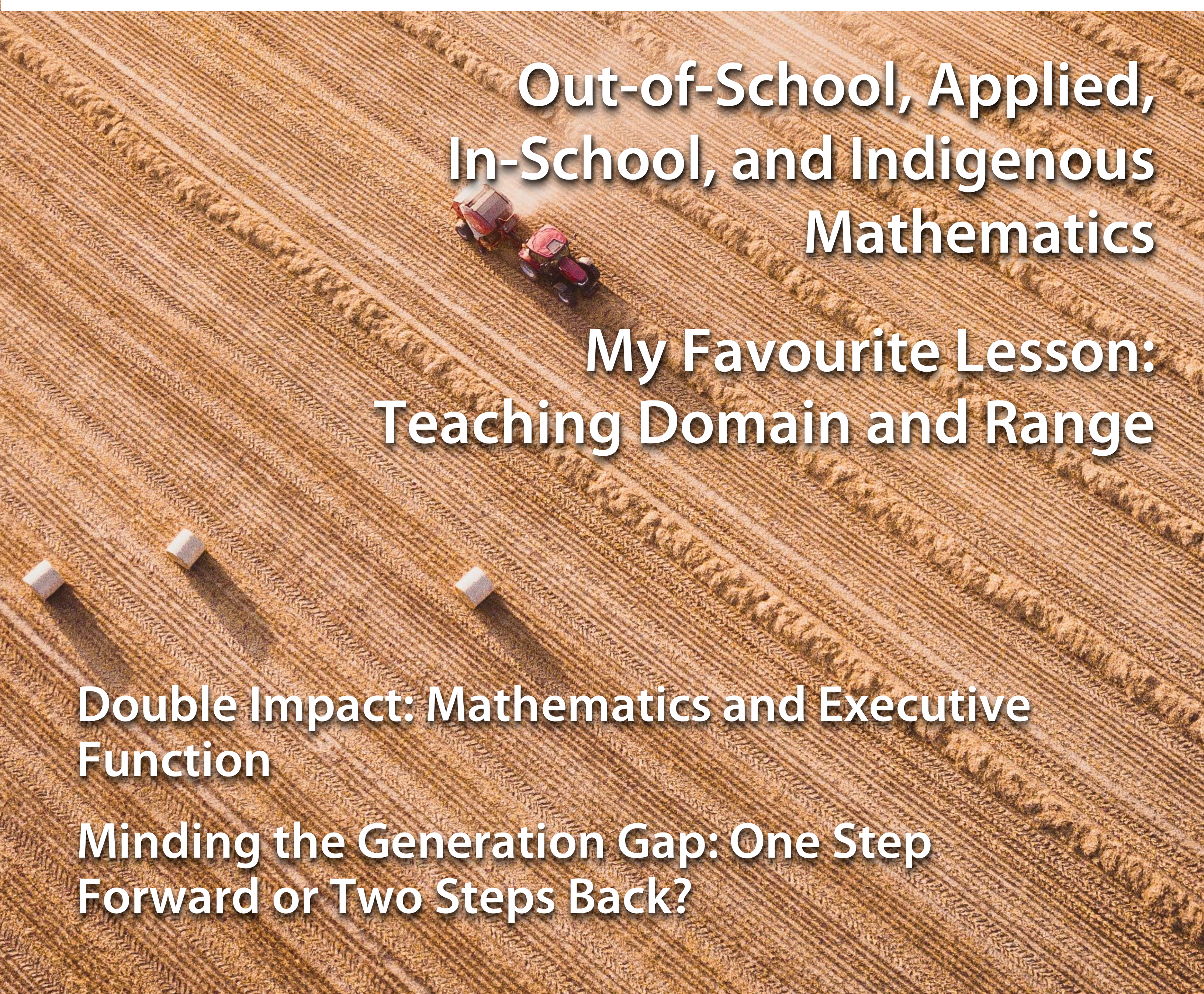

Out-of-School, Applied, In-School, and Indigenous Mathematics

My Favourite Lesson: Teaching Domain and Range

Double Impact: Mathematics and Executive Function

Minding the Generation Gap: One Step Forward or Two Steps Back?

Saskatchewan Mathematics Teachers' Society

AFFILIATE
NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

Recipient of the National Council of Teachers of Mathematics Affiliate Publication Award (2019)

Cover Image

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SMTS Executive (2020-2022)

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Notice to Contributors

The Variable welcomes a variety of submissions for consideration from all members of the mathematics education community, including classroom teachers, consultants, teacher educators, researchers, and students of all ages. Submitted material is assessed for interest, presentation, and suitability for the target audience.

Please submit articles by email to thevariable@smts.ca in Microsoft Word format. Authors should aim to limit submissions to 3000 words or less and include a photo and a short biographical statement of 75 words or less with their submission. Accepted contributions are subject to revision. Editors reserve the right to edit manuscripts for clarity, brevity, grammar, and style.

Editorial

Welcome to yet another pandemic issue of *The Variable*. At this point, it feels like we've crossed most of the pandemic banter off the list in our previous pandemic-themed editorials. There are only so many jokes to be made about Zoom meetings, and only a limited number of ways to offer distanced commiseration for the oscillating feelings of helplessness and cautious optimism. However, if the pandemic response has taught us anything, it is that things don't move forward without tough decisions, and we've made one that we'd like to share with you.

Six years ago, *The Variable* was born with the goal of providing a place for Saskatchewan teachers to gather around ideas for improving the teaching and learning of mathematics. Now, anyone who has endured a high school breakup, knows that prefacing this paragraph with a sentence like that means something is coming to an end, but for those of you who haven't tasted the bitter pill of pubescent heartbreak, we should be clear: This is the last edition of *The Variable*.

Over the course of its publication, the structure of the periodical lived up to its name, changing several times in format, contents, and publication schedule. These subtle changes were always attempts to mirror the realities of teachers, and the decision to move the SMTS communications to a new format was made in the exact same spirit. As with any new chapter, it comes with its share of nostalgia. We are deeply thankful to the contributors, the regular columnists, the SMTS, the NCTM for honouring *The Variable* with the 2019 Affiliate Publication Award, and—most importantly—to the readers.

The end of *The Variable* does not mean end of our mission. Stay tuned for a new format of communications from the SMTS that will be more timely and aligned with the needs of teachers in our province. It has been a pleasure to represent the mathematics teachers of Saskatchewan with each edition of *The Variable*, and we hope you see continued benefits from the future of SMTS communications.

Nat & Ilona,
Co-Editors

Teaching Domain and Range of a Function Through a Constructivist Lens

Jeff Irvine

When I was a beginning teacher, my lesson on domain and range of a function would go something like this:

1. Give definitions of domain and range for students to copy.
2. Work through some examples with the whole class.
3. Assign questions for the students to work on in class and complete for homework.

What were the issues with this approach? Let me count the ways:

- Many of my students developed an imperfect or vague idea of the concepts of domain and range.
- Many could not successfully complete the assigned questions.
- The concepts were often not retained, and so had to be retaught later.
- The presentations were boring and unmotivating for many students.
- This instructional strategy ignored research on productive ways for students to practice new skills.
- The presentations ignored the large volume of evidence that students best acquire concepts when they can connect them to prior knowledge and work collaboratively with other students.

Impact of Student Attitudes and Engagement on Mathematics Achievement

Motivation, engagement, and attitude are significant issues in the teaching and learning of mathematics (Irvine, 2020). Many studies have shown that it is critical to address student motivation in mathematics teaching (Collie & Martin, 2017; Conner & Pope, 2013; Harlow, DeBacker, & Crowson, 2011; Li & Lerner, 2013; Ouweneel, Schaufeli, & LeBlanc, 2013) and that engagement and attitudes are correlated with student achievement (Bodovski & Farkas, 2007; Moller, et al., 2014). Shernoff et al. (2003), for example, found very low levels of engagement in mathematics, with students reporting being more negative and less engaged in mathematics class than any other subject. This is a concern that needs to be addressed, given that many studies also report correlations between engagement and achievement in mathematics. Engagement has been positively linked to perceptions of mathematics (Fung et al., 2018), attitudes towards mathematics (Bodovski & Farkas, 2007), student agency in mathematics (Collie & Martin, 2017), student graduation rates, and students pursuing higher education (Bodovski & Farkas, 2007). However, engagement is also recognized as an important outcome in its own right (Collie & Martin, 2017).

Impact of Instructional Practices on Mathematics Achievement

Research on the effects of different types of practice activities in learning mathematics have found that concepts are better understood and retained longer if the practice is spaced over time. So, instead of having students complete a large block of questions on the same topic for homework, better results are obtained if the questions are spaced out over several days, or even weeks (Hopkins et al., 2016; Peterson-Brown et al., 2019; Rohrer, 2009). There is also ample research evidence (e.g., Moyer et al., 2018) that students learn concepts better when the concepts are self-generated by the students, based on their own prior knowledge; and that student social interactions, such as working in groups, enhance student understanding. The Saskatchewan curriculum documents recognize this by advocating a blend of teacher-led and student-constructed knowledge in mathematics:

> "Teaching mathematics for deep understanding involves two processes: teachers covering content and students discovering content. Content that can and should be discovered by students is that which can be constructed by students based on prior mathematical knowledge." (p.16)

The theory that students build new knowledge based on their current level of knowledge is called constructivism. Constructivism-informed teaching, therefore, focuses on actively involving students in their learning and encourages social interaction among students during the learning process. The teacher's role is that of a facilitator, supporting students by providing activities that allow them to construct knowledge through reasoning, reflection, and social interactions. Classrooms that support constructivism typically involve many group activities, student-generated questions, student choice, and abundant discussions about mathematics concepts. These classrooms are sometimes referred to as math talk learning communities (Irvine, 2017) due to the frequency of learning through discussion. For a summary of constructivism and constructivism-informed teaching, see McLeod (2019).

The following sections offer a series of learning activities, informed by constructivist theory, designed to foster collaboration and sponsor motivation in math class. While they can be used with a variety of concepts, this article illustrates how they can be used to develop students' understanding of domain and range. None of the activities are intended to occupy a full period, and are recommended to be spread out over multiple days or weeks as new topics are introduced. They can even be repeated in different courses, as students encounter new functions. The first two activities should be done sequentially, while the other activities may be done in any order; however, the last two activities are especially useful for review or as preparation for an assessment.

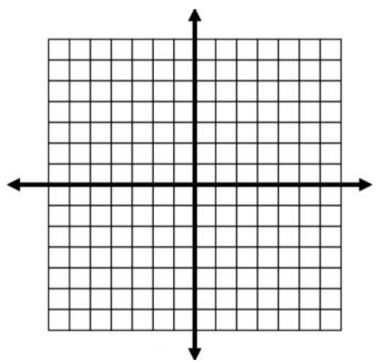
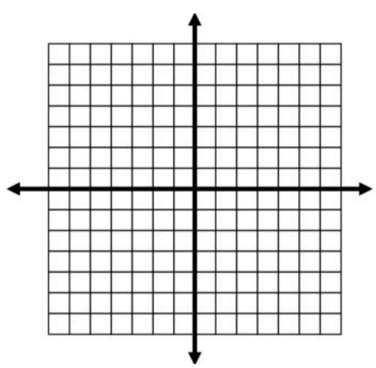
Concept Attainment: Introduction to Domain and Range

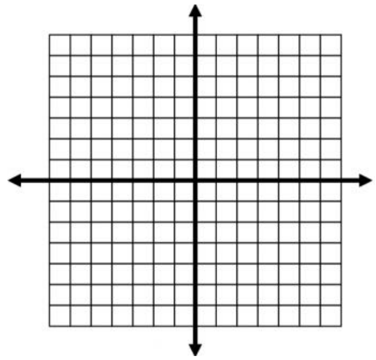
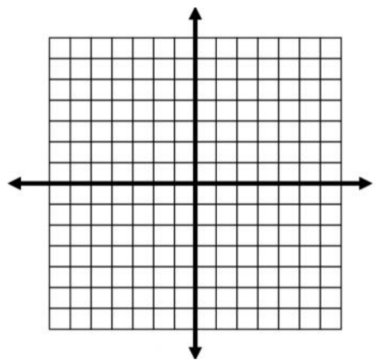
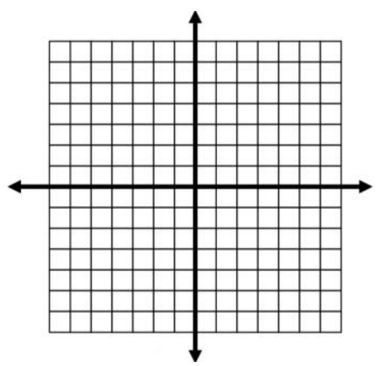
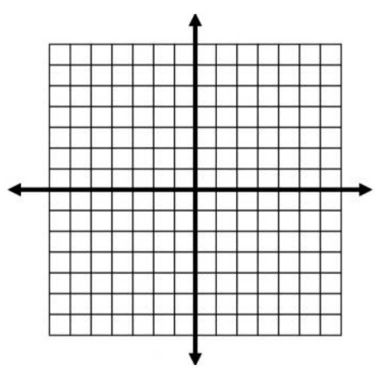
In this activity, students are asked to identify similarities and differences between two columns of information. By identifying similar attributes of the items in a column, students form tentative understandings about the concept illustrated in the examples. They then summarize and draw conclusions about the concept. Table 1 below illustrates the use of concept attainment to introduce domain and range of a function.

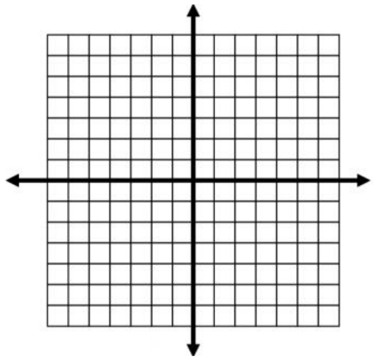
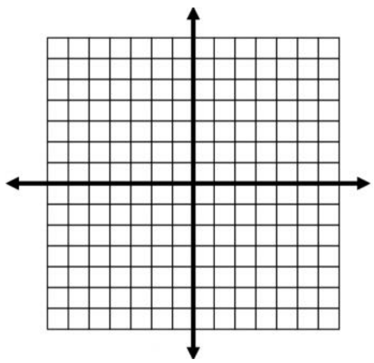
This activity can be done individually, in pairs, or in groups. The intention is that students identify domain as admissible x values (independent variable); range as possible y values (dependent variable); and that the graph of the function is useful in identifying domain and range of a function. While domain and range can be expressed using set builder notation, I have found that it is clearer when introducing the concept to use ordinary language and notation. After this activity, students are asked to generate definitions of domain and range.

Concept Attainment: Domain and Range

1. With or without using technology, sketch a graph of each function in the second column of the table below.
2. What do the entries in the third column have in common? Share your thinking with your partner.
3. What do the entries in the fourth column have in common? Share your thinking with your partner.
4. How are the entries in columns three and four related to the equations and their graphs (columns 1 and 2)? Share your thinking with your partner.

a)	$y = (x-3)^2 + 5$		x can be any real number	$y \geq 5$
b)	$y = 3x + 5$		x can be any real number	y can be any real number

c)	$y = \sqrt{x-3}$		$x \geq 3$	$y \geq 0$
d)	$y = -\frac{1}{2}(x+1)^2 + 3$		x can be any real number	$y \geq 3$
e)	$y = -2$		x can be any real number	$y = -2$
f)	$y = \sqrt{5-x}$		$x \leq 5$	$y \geq 0$

g)	$y = \frac{1}{x-3}$		$x \neq 3$	$y \neq 0$
h)	$y = \frac{-2}{(x-1)(x+4)}$		$x \neq 1$ $x \neq -4$	$y \neq 0$

After you've discussed your responses to questions 1-4 with your partner, answer the following questions.

5. The entries in column two are called the ***domain*** of a function. Write a definition or description of the ***domain*** of a function.
6. The entries in column three are called the ***range*** of a function. Write a definition or description of the ***range*** of a function.
7. How does the graph of a function help in identifying its domain and range?

Placemat: Consolidating Concepts and Addressing Misconceptions

This activity is intended to be completed in groups of 3-4 following the concept attainment activity. Each group is given two "placemats" (see Figure 1) on large paper (e.g., 11x17), one each for the concepts of domain and range. Each student enters their definition of the given concept in their quadrant. These definitions are discussed within the group, after which the group comes to an agreed definition that is then written in the oval. The group discussion facilitates reflection, correction of misconceptions, and consolidation of the concept under study.

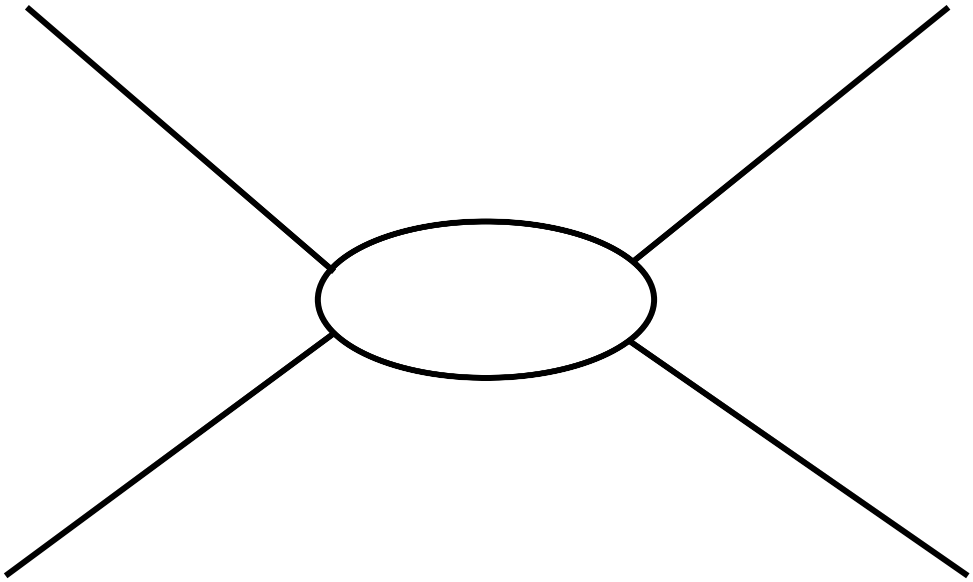


Figure 1. Placemat for consolidating concepts of domain and range

A graph of a particular function may facilitate the discussion of the concepts of domain and range; an example is given in Figure 2.

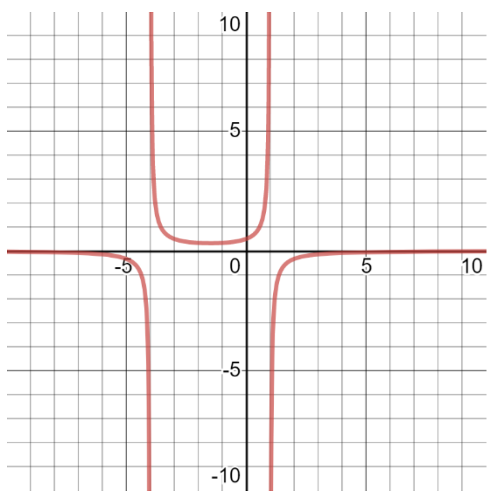


Figure 2. Graph of $y = \frac{-2}{(x-1)(x+4)}$

Following the placemat activity, a whole-class summary should consolidate the following ideas:

- the graphs of the functions assist in identifying domain and range;
- for domain, values of x that would cause problems are excluded (e.g., values that result in square roots of negative numbers);

- for range, possible y values resulting from problem x values are excluded.

Emphasis should be placed on reasoning and conjectures. It is suggested that students then proceed to the first round of jigsaw, as described in the next section.

Jigsaw Domain and Range for Different Families of Functions

The jigsaw strategy allows students to build their knowledge while supported by the members of their groups. Like the other activities, the jigsaw routine invites reflection and clarification of misconceptions through social interaction. Students become the experts on their functions through discussion in a peer group, then share their expertise with the members of their home group. The activity proceeds as follows:

1. Students start in home groups. All groups receive the same set of functions; each student in the group receives a different function. It can be helpful to write the functions on individual slips of paper and colour-code them (see Figure 3).
2. Students go to expert subgroups, which consist of all students with the same function (or colour, if colour-coded; see Figure 3). In their subgroups, students graph their functions and identify their domain and range.
3. Students return to their home groups and take turns sharing their graphs. They invite the other members of their home group to identify the function's domain and range, offering support as necessary.

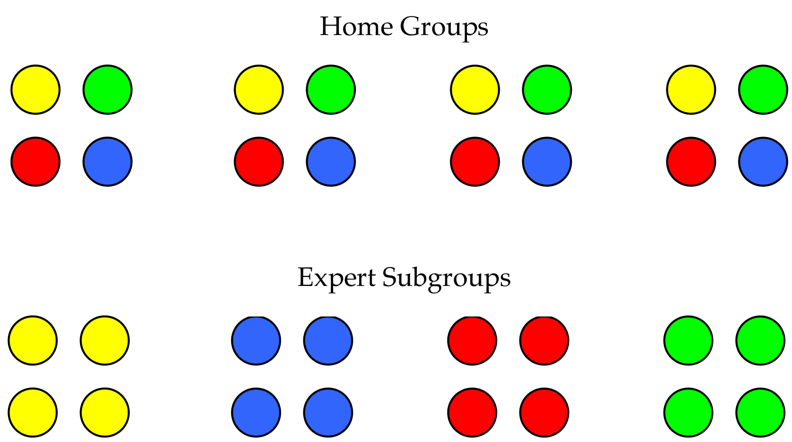


Figure 3. Jigsaw

Suggested Functions for Jigsaw Round 1

a) $y = -2(x-3)^2 + 2\ ;\ y = -2x + 3$
b) $y = 3(x+1)^2 - 2\ ;\ y = 4$
c) $y = -(x+2)^2 - 4\ ;\ y = \frac{3}{5}x - 1$
d) $y = (x-3)^2 + 2;\ x = -5$
e) $y = \frac{1}{2}(x-1)^2 + \frac{1}{3};\ x + y = 10$

The set of functions above is familiar to students in Pre-Calculus 20 and Foundations 20, and thus the concepts of domain and range are reinforced by building on prior knowledge. In keeping with the research on spaced practice, other rounds of jigsaw are saved until appropriate additional functions have been learned. Suggested functions for additional rounds of jigsaw for Pre-Calculus 20 and 30, and Foundations 30, can be found in Appendix A.

Gallery Walk: Consolidating Understanding

Gallery walks involve students first completing an activity within their group, then analyzing the work of other groups. By moving among the solutions as a group, students are socially supported and thoughtful questions often arise.

This activity begins with each group randomly selecting two functions from a pile of functions written on slips of paper. Suggested functions are given in Appendix B. The group sketches the functions and describes their domain and range on large chart paper or on a whiteboard, which is then posted on the wall to facilitate viewing (see Figure 4). Each group moves to the next piece of chart paper, and verifies through discussion whether the graph, domain, and range of the functions are correct. If errors are found, they should be corrected on the chart paper by the group that finds them. The groups then move to the next chart paper and repeat the activity. It is important that each group finish by returning to their own chart paper to check their work for accuracy.

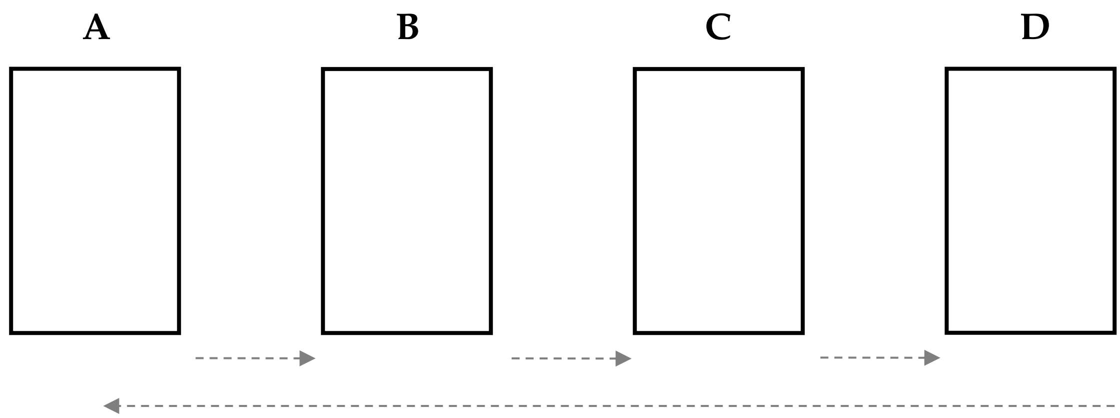


Figure 4. Gallery walk

Inside/Outside Circle: Confirming Understanding

This activity is excellent strategy for reviewing concepts. At the outset, each student writes a function and sketches its graph on one side of an index card. On the back of the card, the student indicates the domain and range of the function. Students then proceed as follows:

1. Half of the students form a small circle, facing outward.
2. The other half of the students form another circle around the first circle, facing inwards so that every student is facing a partner (see Figure 5).
3. The student on the inside circle holds up their index card, showing only the equation/graph side. The student on the outside circle identifies its domain and range (the inside student may offer prompts, such as "Are there any restrictions?" and so on).
4. Once the domain and range have been correctly identified by the outside student, they hold up their own index card and repeat step 3, with the roles reversed.

5. Once each pair has finished, the outside circle rotates one student to the right. The new student pairs repeat steps 3 and 4. Repeat until students return to their original pairs, or as time permits.

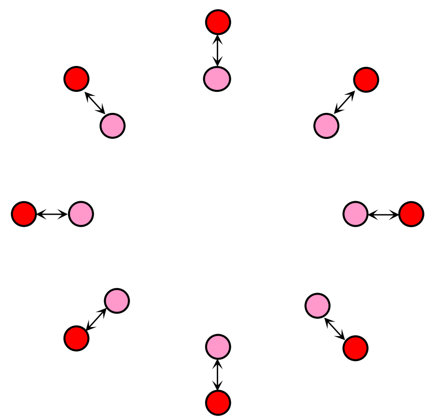


Figure 5. Inside/outside circle

I have found that the activities described above have resulted in students being more motivated and engaged in learning mathematics. In addition, when students take an active role in constructing their own learning, they feel a sense of agency and ownership. The consequences of this agency are more involvement, deeper thinking about the mathematics content, and greater comprehension and retention. This is supported by research that has shown that teaching with a constructivist stance is related to increased student engagement, persistence, retention, and achievement (Smith & Star, 2007).

Instructional strategies for any topic can be adapted to sponsor student-centered learning, using strategies such as the ones outlined in this article. While initially this may take some time, the outcomes with respect to student understanding and retention are well worth the effort, and the constructivist stance will thus become an important aspect of your instructional repertoire.

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Appendix A: Suggested Functions for Additional Rounds of Jigsaw

Jigsaw Round 2

a) $y = \sqrt{4 - x}\,;\ y = \frac{1}{(x-2)(x+3)}$
b) $y = \sqrt{x + 3}\,;\ y = \frac{x-2}{(x+1)(x-3)}$
c) $y = \frac{3}{(x+2)(x-1)}\,;\ y = \sqrt{x + 10}$
d) $y = \sqrt{x - 4}\,;\ \ y = \frac{1}{(x^2-1)(x+4)}$
e) $y = \sqrt{9 - x^2}\,;\ y = \frac{1}{x^2-4}$
f) $y = \sqrt{x^2 + 9}\,;\ y = \frac{1}{x^2+4}$

Jigsaw Round 3

a) $y = 3^x - 4\,;\ y = \log{(x + 2)}$
b) $y = -2^x - 3\,;\ y = 2\log{(x - 4)}$
c) $y = 3^x + 3\,;\ y = \log\ x$
d) $y = 4^x - 2\,;\ y = -2\log{(x + 1)}$
e) $y = 2^x + 1\,;\ y = \log{(x - 3)}$

Jigsaw Round 4

a) $y = \sin x\ ;\ y = -\tan x$
b) $y = 3\sin x\ ;\ y = -\tan x - 3$
c) $y = \cos x\ ;\ y = -3\cos x - 2$
d) $y = \tan x\ ;\ y = -2\sin x + 1$
e) $y = -\cos x + 2\,;\ y = \frac{1}{2}\sin x + 1$

Appendix B: Suggested Functions for Gallery Walk Activity

a) $y = \frac{5}{(x-2)(x+3)}$
b) $y = \frac{(x-2)(x-3)}{(x+1)(x+4)}$
c) $y = \frac{x^2-9}{x^2+9}$
d) $y = \frac{4}{x^2+9}$
e) $y = \frac{-2}{x^2-9}$
f) $y = 2\sqrt{x^2 + 16}$
g) $y = -\frac{1}{2}\sqrt{x^2 - 4}$
h) $y = \frac{2}{5}\sqrt{x + 2}$
i) $y = -\frac{3}{4}\sqrt{x - 5}$
j) $y = -\sqrt{9 - x^2}$

Jeff Irvine, PhD, has been a secondary mathematics teacher, department head, and vice principal. He has taught at three faculties of education and at a community college. For several years, he was an Education Officer in the Curriculum and Assessment Policy Branch of the Ontario Ministry of Education, where his portfolio was grades 7 to 12 mathematics for the Province of Ontario. Jeff is coauthor or contributing author for 11 high school mathematics textbooks. With over 45 years in education, Jeff is particularly interested in the role of student motivation in mathematics achievement.

Alternate Angles

Alternate Angles is a column on problems from multiple perspectives: various methods that could be used to solve them, insights we get from their solution, the new paths that they can lead us to once they have been solved, and how they can be used in the classroom.

Problematic Probabilities

Shawn Godin

Welcome back, problem solvers! I hope the upcoming school year will be a great one for you and your students. I have recently retired and am hoping to add a new feature to the column. Occasionally, I will make use of technology, and will make the resulting files available to readers through a shared folder in my Google Drive. You can access this Drive at https://bit.ly/3gglgQe. For this column, I have shared the spreadsheet that I reference below. As well, I have made a short video on how I created the spreadsheet for those who would like a tutorial. I hope to work my way back through my past columns and add similar “treats” to complement them. I hope that you find these supplementary materials useful.

Last issue, I left you with the following problem:

> Alice places a coin on a table, heads up, then turns off the light and leaves the room. Bill enters the room with two coins, puts them onto the table, then leaves. Carl enters the dark room and removes a coin at random. Alice re-enters the room, turns on the light, and notices that both coins are heads up. What is the probability that the coin Carl removed was also heads up?

This was problem B2 from the 2020 Canadian Open Mathematics Challenge run by the Canadian Mathematical Society. The Canadian Mathematical Society runs several mathematics competitions each year. As well, they have resources for students and teachers. One of the resources is the free online problem-solving journal *Crux Mathematicorum*, which features a section, MathemAttic, aimed at pre-university students and their teachers. You can check out the resources offered by the CMS on their website at https://cms.math.ca.

Probability problems are interesting: We often deal with situations that involve chance and, in many cases, we think we have a good intuition for how things will turn out. However, our intuition often fails us.

Let's start by looking at a classic problem. I have three coins and propose we play a game. One coin has both sides marked heads, one coin has both sides marked tails, and one coin has one side marked heads and one side marked tails. All the "heads" markings are indistinguishable, as are all the tails markings. The three coins are put into a bag. You mix them up, reach in, pull one out, and place it on the table, revealing one of the faces. I propose the following wager: I will guess what the other side is, and if I guess correctly, you'll give me $1. If I'm wrong, I'll give you $1. Do you want to play?

Most people will assume that the chances of me picking correctly are 50%—and they would be wrong! Don't believe me? The beauty of many probability problems is that you can simulate them by doing an experiment. To simulate this problem, cut three circles out of cardboard. Mark one HH, one HT, and one TT. Put them in a container and pull one out, reveling only one side. My strategy is that I will always guess whatever side is showing. So, if you pull out a coin and it shows heads, I will guess heads. If you run the experiment for enough trials, you will see that my winning percentage is closer to $66\frac{2}{3}\%$ than to 50%. Why does this happen?

Initially, we may decide there are only four possible outcomes: the HH coin is picked and I win, the TT coin is picked and I win, the HT coin is picked with H showing and I lose, and the HT coin is picked with T showing and I lose. Another faulty way to look at things is that if I see H on the coin, it can either be the HH coin and I win, or the HT coin and I lose. Both lines of thinking suggest I should win only 50% of the time, contradicting the experiment we've just run. What's wrong with this analysis?

One of the places that people fall into trouble is with the definition of the probability of an event, A. We use the well-known formula

$$P(A) = \frac{n(A)}{n(S)},$$

where $P(A)$ is the probability of event A occurring, $n(A)$ is the number of ways that A can occur and $n(S)$ is the total number of possible outcomes to our "experiment." However, this formula only holds if the outcomes we are counting are all equally likely. For example, if we roll two dice, the sum can be any number from 2 through 12. However, the probability of rolling a 7 is not $\frac{1}{11}$ (there are 11 outcomes, one of which is 7), because each of these results are not equally likely. The values 1 through 6 are equally likely on each die, so we can break our experiment down into 36 possible outcomes, as shown in the table below.

	1	**2**	**3**	**4**	**5**	**6**
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

From the table, we can see that 7 is the most likely outcome, occurring 6 out of 36 times, so the probability of rolling a 7 is in fact $\frac{1}{6}$ (which is considerably greater than $\frac{1}{11}$!).

This trap is similar to the one we fell into with the coins problem. If we label our two-headed and two-tailed coins as H_1H_2 and T_1T_2, we can see things more clearly. In fact, there are six possible outcomes in our experiment, because either side of the three coins may be turned up. If we look at all of these equally likely possibilities, we will be able to accurately determine the probability that I guess correctly.

Coin	**Side Showing**	**My Guess**	**Correct?**
H_1H_2	H_1	H	Yes
H_1H_2	H_2	H	Yes
HT	H	H	No
HT	T	T	No
T_1T_2	T_1	T	Yes
T_1T_2	T_2	T	Yes

Now, we can see that the wager I proposed is unfair, since I will win $\frac{2}{3}$ of the time. The table also allows us to see it another way: I will win if the HH or TT coin is chosen and lose if the HT coin is chosen. (You may want to investigate whether there is a way that we can adjust the payoff scheme to make the game fair.)

Another classic probability problem is the Monty Hall problem, named after the host of the game show *Let's Make a Deal*. In the final deal of the show, one of the contestants is shown three doors. Behind one door is a prize, while the others usually contain joke prizes; only the host knows the location of the prize. The contestant picks one of the doors, but instead of opening the chosen door, the host first opens a door behind which there is a joke prize. The contestant is then given a choice: Keep your door or switch. Again, our instincts tell us that since there are two doors, you must have a 50% chance of winning either way. But again, our instincts would be wrong. The problem comes down to this: If you keep your original choice, you will win only if you picked the winner in the first place. This would happen $\frac{1}{3}$ of the time. However, by allowing you to switch after showing you one of the duds, Monty is really giving you the choice to keep your original door or switch and take the contents of the other *two* doors.

This is another fun problem to simulate with an in-class experiment. There are also many simulation websites that will allow you to participate in the game and keep track of your wins and losses. Some sites will even let you run the simulation for many cases where you decided to either always switch or always keep your original choice. The website www.math.ucsd.edu/~crypto/Monty/monty.html, for example, has two versions of the problem programmed: the classic case, where Monty knows where the prize is, and a case where Monty doesn't know and sometimes reveals the prize, so you have to play again. I will leave it to you to determine what you should do in the case where Monty doesn't know what is going on.

Now, back to our original problem. Again, a simulation is in order. The figure below shows a simulation of the situation done in Google Sheets. Notice that in some cases, if we do things randomly, there will not be two heads left in the room. We are dealing with *conditional* probability in this case. This means that, since we know that Alice sees two heads, we ignore all of the cases where this doesn't happen. In the diagram, we see that after 10 trials, the conditions of the problem have been satisfied 5 times, with heads showing up 2 of those 5 times. In the last few columns, we keep track of the Total Cases (the number of times HH was left in the room), the number of Favourable Events (the number of times HH was left and Carl took H) and the Probability (experimental probability as a percentage, $\frac{\text{Events}}{\text{Total Cases}} \times 100$).

Alice (1)	Bill 1 (2)	Bill 2 (3)	Carl remove	Left	Satisfy	Carl's coin	Total Cases	Events	Prob	☐
							0	0		
H	H	H	3	HH	TRUE	H	1	1	100	
H	H	T	3	HH	TRUE	T	2	1	50	
H	T	T	3	HT	FALSE	T	2	1	50	
H	H	T	2	HT	FALSE	H	2	1	50	
H	T	H	3	HT	FALSE	H	2	1	50	
H	H	H	1	HH	TRUE	H	3	2	66.6666666	
H	T	H	2	HH	TRUE	T	4	2	50	
H	H	T	3	HH	TRUE	T	5	2	40	
H	T	H	1	TH	FALSE	H	5	2	40	
H	T	T	1	TT	FALSE	H	5	2	40	

You would probably want to run the simulation quite a few more times to feel more comfortable with the results. The spreadsheet, as well as a short video explaining how I created it, are available on my shared Google Drive folder (https://bit.ly/3gglgQe). Note that with spreadsheets, when a random number is being used, any change in the spreadsheet causes all of the random numbers get recalculated. I put a checkbox at the top of the spreadsheet to allow me to quickly recalculate all the random numbers. This way, I can look at 100 cases, for example, see what the result is, then recalculate and see how the result changes.

To find the theoretical result, we must look at all possible cases, as shown in the table below. There are four possibilities for the three coins, based on the four possible outcomes for Bill: HH, HT, TH, TT. For each of these, there are three possible coins that Carl could remove: Alice's, Bill's first or Bill's second. When all cases are looked at, there are only 5 where HH is left over. In three of those cases, shaded green, Carl removed a head, in the other two, shaded red, he removed a tail. We can conclude that the probability of Carl removing a head is $\frac{3}{5}$, or 60%.

Alice	**Bill**	**Carl**	**Left**	**Conditions?**
H	HH	H	HH	yes
		H	HH	yes
		H	HH	yes
H	HT	H	HT	no
Alice	**Bill**	**Carl**	**Left**	**Conditions?**

		H	HT	no
		T	HH	yes
H	TH	H	TH	no
		T	HH	yes
		H	HT	no
H	TT	H	TT	no
		T	HT	no
		T	HT	no

As these examples show, when it comes to calculating probabilities, we need to proceed with caution. In particular, we must always be careful to ensure that we are dealing with equally likely situations. And although our initial instincts may lead us astray, simulations can help us gain some insight into the situation.

I want to thank the editors of *The Variable* for inviting me to contribute to the journal and for their continued support and encouragement. I look forward to contributing, in some form or another, to the forthcoming newsletter. Have a great school year, and happy problem solving!

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Double Impact: Mathematics and Executive Function[1]

Candace Joswick, Douglas H. Clements, Julie Sarama, Holland W. Banse, & Crystal A. Day-Hess

The teacher displayed counting cards that included both dots and numerals in order from one to five, as she counted them with her students. She then turned the cards facedown, keeping them in order, and began an identify-a-hidden-card activity with the class.

Teacher: Pete, can you point to a card, please?
Petrov: [Points to the second card from the left]
Teacher: Thank you. I know that is card "two"! Pete, can you turn it over to show the class?
Class: [As Pete shows the card] Two! It's two!
William: How'd you do that?
Petrov: She counted in her head. One [pointing to the first card], two [holding the "two" card as high as he can].
Teacher: Amelia, what card is Pete holding up?
Amelia: Two.
Teacher: How do you know it is two?
Amelia: I see two dots [on the card].
Teacher: How else can you tell that this is card two? Naomi?
Naomi: There's a two on the card.

This class was engaged in the third of three card activities that develop number sense and number skills. In this article, we describe how we have used and modified this activity both to develop mathematical competencies and to develop important higher-order, or executive function, skills. We conclude by providing strategies for modifying any mathematics activity to similarly get "double impact"—simultaneous development of young children's mathematical proficiencies and executive function skills.

Prekindergarten students enjoyed the card activity and pretended they had X-ray vision

Two activities to develop executive function skills

To begin, consider the first two activities: ordering cards and identifying the missing card. The Order Cards activity asks children to arrange a set of numeral cards in a row, 1–5 or 1–10 (all activities are from Clements & Sarama, 2013). The What's the Missing Card? activity extends the Order Cards activity by asking children to identify a card that has been removed from the row. Such activities engender the number concepts of ordinal counting, numeral recognition, number order, and the successor, or number-after, principle that are essential learning in

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Four-year-old prekindergartner Violet first ordered a shuffled stack of numeral cards and then counted them to ensure they were arranged correctly

Violet and her teacher turned the ten cards facedown

early childhood mathematics but often are not developed in depth (Clements & Sarama, 2014; Sarama & Clements, 2009).

So far, so good. However, despite their moderate to high mathematical demands, neither activity does much to help children develop other executive function (EF) skills. What are these EF skills? How we can develop both high mathematics and executive function competencies by modifying activities?

EF processes allow people to control, supervise, or regulate their own thinking and behavior and are critical for young children's learning (Clements, Sarama, & Germeroth, 2016). For example, EF predicts math achievement as well as success in school broadly (Clements, Sarama, & Germeroth, 2016). Most teachers rate such EF components as inhibition and attention shifting as important for math thinking and learning, and these ratings increase with teaching experience (Gilmore, 2014). Some researchers argue that EF processes constitute “a major characteristic of productive mathematics learning” (De Corte et al., 2011, p. 155). Interestingly, many studies show that EF is associated more highly with mathematics than literacy or language (e.g., Blair et al., 2015; see a review in Clements, Sarama, & Germeroth, 2016; McClelland et al., 2014).

EF includes three categories: (1) inhibitory control, (2) working memory, and (3) attention shifting and cognitive flexibility. Inhibitory control allows one to keep from acting impulsively. Consider the following problem:

> There were six birds in a tree. Three birds already flew away. How many birds were there before some flew away?

Children must inhibit the immediate desire to subtract engendered by the phrase flew away and instead calculate the sum. Working memory allows one to both hold information in short-term memory and process that information. Children solving a measurement problem may have to keep the problem situation and their solution in mind while they perform a necessary computation, interpret the result of the computation in terms of the measurement units, and then apply that to the problem context to solve the problem. The third category, attention shifting and cognitive flexibility, includes two closely related EF processes that are considered simultaneously. They allow one to switch attention as a situation requires and be flexible in thinking. For example, children may have to count meters and then centimeters as part of a meter or abandon a “rule” they determined in a Guess My Rule game when a new example is inconsistent with their original thinking.

Despite the importance of EF, efforts to help children develop it—and those with fewer home and community resources especially need that help—have, at best, mixed results. Computer games and other direct training approaches of EF have been moderately successful in only some studies, and the effects seem limited to those specific contexts (Clements, Sarama, & Germeroth, 2016). Fortunately, a bidirectional relationship between math and EF appears to exist—the development of one seemingly promotes the development of the other (Clements, Sarama, & Germeroth, 2016). So, because developing both EF processes and mathematical proficiencies is essential for young children, high-quality mathematics education may have the dual benefit of not only teaching this important content area but also developing young children's EF processes, using precious instructional time wisely.

The X-ray Vision activity

That brings us back to the third activity with cards, identify a *hidden* card, which was designed to include higher EF demands. In the X-ray Vision activity (Clements & Sarama, 2013), cards are arranged in numerical order and then turned facedown on the table. One child points to one of the cards, and a partner must use his or her "X-ray vision" to name the card before turning it over to confirm. Then the card is turned back to facedown, and the children switch roles. Of course, children actually must use mathematical processes to determine the identity of the hidden card. (But they still think they are super heroes when they can do it!) The mathematical demands of this activity can be differentiated: Children may count from one, touching each card, until they arrive at the selected card, or count down from the highest card to the selected card.

The teacher and child played a turn of X-ray Vision, the teacher in the role of selector *and the child in the role of* identifier

A second version

In a second version of the X-ray Vision activity, children leave already-identified cards faceup. This encourages students to count on or count down from those cards to the chosen card or to identify a number between two faceup cards (e.g., 4 and 6). This type of Identify-a-Hidden-Card number activity has a high mathematical demand for young children: They must count, using one-to-one correspondence, forward or backward, from one number to another.


Before the next round of the game, the teacher reminded Violet where the number-1 card was and then pointed to the fourth card in line

Further, the X-ray Vision activity also has high EF demands. The activity engenders each of the three primary categories of EF processes: (1) inhibitory control, (2) working memory, and (3) attention shifting and cognitive flexibility. Violet, a four-year-old prekindergartner, could count ten objects and even produce, or count out, a set of ten items as well as recognize numerals 1–10. She was asked to first order a shuffled stack of numeral cards on the table. Then she and her teacher counted the cards, as they pointed to each, to ensure the cards were arranged correctly before turning them facedown.

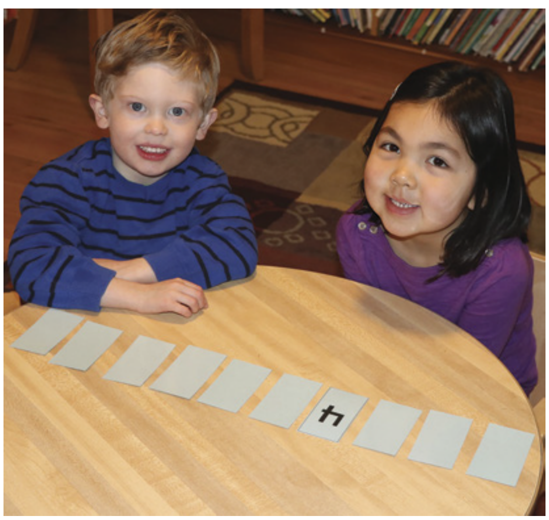
The teacher asked Violet to point to any single card, then the teacher said, “Hmm. I'm using my special X-ray vision trick to figure out what number is on that card. Hmmm. I-t—i-s—f-i-v-e!” The teacher then had Violet turn the card over to reveal that is was indeed a number 5. The child, as one might anticipate, was delighted with her teacher's X-ray vision trick!

The teacher had Violet replace card 5, facedown, in the row of cards on the table. “Now, Violet,” the teacher asked, “If I chose a card, can you name the number on the card without looking?” After Violet excitedly agreed, her teacher reminded her where the number-1 card was and then pointed to the fourth card in line. “Alright, Violet, what is this card?” the teacher asked.

The child paused and then exclaimed “Four!” as she held up four fingers.

“How do you know it is four?” the teacher asked Violet, incredulously.

She answered, “I counted one, two, three, four,” as she looked at, but did not point to, the cards on the table. The teacher then flipped over the selected card and confirmed that Violet had used her own X-ray vision to determine the correct answer.



Violet used her own X-ray vision to correctly determine the number on the card her teacher had chosen

The X-ray Vision activity allows for the incorporation of inhibitory control, working memory, and attention shifting and cognitive flexibility. For instance, in the introduction to this activity, Violet used inhibitory control to keep herself from just making a wild guess or reaching to the selected card and just flipping it over to reveal its identity. She used working memory to sort the cards into numerical order to begin the activity and to keep both the location of the selected card in mind and to remember and apply a process to solve the task. Her working memory and attention shifting processes were further used as she and the teacher exchanged selector and identifier roles with one another, taking turns selecting cards for one another and testing one another's special X-ray vision tricks in subsequent games.

In another game of X-ray Vision, played between children, Cooper selected card 8 for Violet. In the previous turn, card 4 had been identified by Cooper and was left faceup. To identify the card Cooper had just selected for her, Violet first touched card 5, saying “five,” then “six, seven,” as she counted each one. She stopped at “eight,” with her finger on the card. This interaction demonstrates Violet's ability to both count on and employ attention shifting.

When Cooper and Violet played as a pair with the teacher, and card 9 was selected, Violet told Cooper that she knew that “ten was the last, and nine comes before.” In this thinkpairshare version of X-ray Vision, Violet and Cooper agreed on the identity of the hidden card, although Violet articulated her mathematical reasoning in a more sophisticated way than when not given the opportunity to share her reasoning with a peer.

Differentiation

Both the mathematical demands and executive function demands of activities like X-ray Vision can be increased or decreased to accommodate the needs of all learners. For example, to increase the mathematics demand, instead of having a child count from the first card each time, ask students to count down from the highest card or count on or count down from a previously uncovered card. The mathematical demands can be lowered by decreasing the number of cards, such as using only cards 1–5, or by using counting cards with dot arrangements and numerals for children who lack numerical recognition, as was done by the teacher in the opening vignette. Conversely, adding more cards with higher numbers increases mathematical demand.

Many of the modifications used to increase or decrease mathematical demand similarly alter the executive function demands. For example, to increase mathematical demand, the cards need not be arranged in one single horizontal line but can be arranged in an array—cards 1–10 arranged in two rows of five cards, for instance. Such a change increases the demand on a child's use of attention-shifting processes. That is, a child may come to the end of the first row, having counted 1–5, and need to shift attention from counting to making sense of where the next card is placed—as card 6 is now below card 1 and not to the right of card 5. With arrays, some children may be able to subitize the quantity of cards in the first row—say “five”—and begin counting from card 6 in the second row. Counting on from a previously identified card not only increases mathematical demand but also increases the use of inhibitory control processes. Engaging with the same general activity, but with varying structures and emphases or mathematical demand, also increases children's cognitive flexibility. Yet, EF can be increased or decreased without changing the mathematical demand. For example, when choosing a card to identify, children must use working memory to recall which card was selected as they go through the activity process. To decrease the working memory demand, a chip can be used to continuously mark the selected card.

Other ways to decrease the EF demands of X-ray Vision are to decrease various roles that children can play. For example, decrease the EF demands by allowing the child to play only the role of identifier. That is, the teacher will always select the cards, and the child will always name the cards—roles will remain constant. Or increase the EF demands by having children play X-ray Vision in pairs. Using think-pair-share methods, students can confer with each other and then collectively answer which card has been selected. Or have children switch roles: Allow one child to play the selector and one to play the identifier, switching roles at each turn. The demands of each of the EF processes—attention shifting and cognitive flexibility, inhibitory control, and working memory—can be increased or decreased in X-ray Vision. The visual representation shows selected variations of the X-ray Vision activity, suggesting ways to increase or decrease both mathematical and executive function demands.

Many high-quality mathematics activities for young children can be modified to have high EF demands. One might think of varying demands in activities, both mathematical and EF, like differentiation: Demands may be increased or decreased for children to meet their individual needs and the activity goals. See Table 1 for suggestions.

Task modifications

	Mathematics	Attention shifting and cognitive flexibility	Inhibitory control	Working memory
	Provide opportunities and support for children to—			
Principles	*engage in challenging but achievable mathematics activities.*	*switch attention as a situation requires; search for a new strategy if the first one attempted fails.*	*keep from acting impulsively.*	*hold information short term and process or apply such information.*
Suggestions	• Increase or decrease the numbers used. • Change representations (e.g., dot cards or numeral cards) or manipulatives. • Make the activity more or less abstract. • Present oral or written activities. • In general, move up or down levels of a learning trajectory (Clements & Sarama, 2013; Sarama & Clements, 2009).	• Increase or decrease the number of roles a child plays or switches between. • Increase or decrease the number of steps in the activity. • Present the same mathematical activity in many different ways (e.g., X-ray Vision cards in linear arrangement vs. an array). • Use different contexts for a topic such as addition, including multistep problems. • Give problems that require flexible thinking, such as finding all pairs of positive whole numbers that sum to six.	• Increase or decrease the number of turns children take during an activity (e.g., increase or decrease their "wait time"). • Have children work in pairs (e.g., think-pair-share) to discuss ideas. • Use problems with "tricky" phrasing, such as, "There were six birds in a tree. Three birds already flew away. How many birds were there from the start?" • Encourage positive behaviors and attitudes (Fuhs, Farran, & Nesbitt, 2013).	• Increase or decrease the number of and/ or demands of the processes for the activity (e.g., how many things a child needs to remember). • Use visual mediators, such as pictures of each step of the activity, then remove them to increase working memory demands. • Have children explicitly state that they are committing an idea (e.g., a number or a card) to memory. • Increase the use and number of steps in multistep problems.

Table 1: A table shows general principles and suggestions for modifying mathematical and executive function demands in activities for young children.

Mathematical and executive function (EF) demand matrix

Mathematical demands

Executive Function Demands	Low → High	
High	Activities here often sacrifice the mathematics for EF. This is undesirable, but these activities could be used sometimes to develop EF when there is already some mathematics mastery. For example, Simon Says-type activities can be useful to develop EF. Counting can be incorporated, for instance, by calling to children to do actions a particular number of times while counting aloud.	Activities here are preferable. But activities with high EF and high mathematical demands can often be very challenging, so mathematics and EF should be supported until children can be given activities with high demands in both. Sometimes activities here may be given as a challenge when appropriate.
Low	Activities that are low in both mathematical demands and EF demands can be great for fun or transition time. These activities can be easy to enact and are engaging for children, but will not necessarily increase their mathematical or EF skills. Time on these activities should be limited or minimized so as to avoid detracting from time spent on more demanding activities.	Activities here are acceptable and common. That is, many mathematics activities have high mathematical demands but do not have a focus on building children's EF skills. Thus, these activities should be used but can be modified to increase EF demands.

Table 2: Activities should be selected on the basis of where children are mathematically and the development of their executive function skills.

Designing for optimal learning and problem solving

Asking young children to face high mathematical demands and high EF demands may be overwhelming and inappropriate. So, what is best? To begin, very low mathematics demands are usually undesirable—the mathematics should not be sacrificed for high EF demands unless the activity goal is EF training only, and activities with both low EF and low mathematical demands should be used cautiously (see Table 2). If they are necessary for some children, those children should subsequently receive additional experiences that slowly but surely increase the demands so they build both competencies, catching up to their peers.

Activities with moderate to high mathematical demands that do not explicitly or intentionally support or demand the use of EF processes are, of course, useful (see Table 2). If incorporating moderate to high EF demands into activities with high mathematical demands proves too difficult for students, some folding back to lower EF or mathematical demands is appropriate before returning to higher demands. Ultimately, activities with both high EF and mathematics demands are most desirable, but not all children can nor should start there or always be there. Some lowering of either EF or mathematical demands allows for the other area to be emphasized and for some direct instruction or support on EF or mathematics as necessary.

Studies have shown that children's EF processes can be trained and increased through particular curricula and programs that target both content and EF simultaneously (e.g., Bierman et al., 2008; Clements, Sarama, & Germeroth, 2016; Clements et al., forthcoming; Raver et al., 2011; Riggs et al., 2006; Weiland et al., 2013). EF may be developed in learning the mathematics in the context of challenging activities, not in "exercising" the mathematics once learned (Clements, Sarama, & Germeroth, 2016). And, because both EF processes and subject matter proficiencies are required to support optimal learning and problem solving, designing interventions that interweave the two makes sense. Teachers may consider using the principles and suggestions in Tables 1 and 2 to modify their own activities for increased or decreased mathematical and EF demands, generating a positive double impact on their students' development of mathematics and executive function.

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Out-of-School, Applied, In-School, and Indigenous Mathematics

Glen Aikenhead

Introduction

High school mathematics teacher Mr. Hazelton was lying in his hospital bed, waiting for his nurse to calculate his dose of medication. Feeling apprehensive, he Googled "mathematics proficiencies of nurses" and found several articles of interest. Perlstein and colleagues (1979), for example, discovered that the mean score on standardized test items for 95 practising pediatric nurses was 77% with a range from 45% to 95%. More recently, Brown (2002) reported a mean score of 75% on a standardized mathematics test among 850 nursing students across Canada and the U.S., and Özyazıcıoğlu and colleagues (2018) found that the third-year nursing students in their study scored 79% on ratio and proportion questions. Should Mr. Hazelton be worried, faced with these somewhat discouraging results?

His mathematical mindset made him worry. "This lack of mathematical literacy [by nurses] has often been included as part of the widespread debates regarding the instances of nurse prescription errors, sometimes resulting in serious patient injury or even death" (Jarvis et al., 2015, p. 1). Despite his apprehension, when Mr. Hazelton's medication arrived, he nervously swallowed it. And, to his great relief, everything turned out well. Was he just lucky? It turns out that Mr. Hazelton had no reason to worry. His positive result with the medication was, in fact, the outcome with the highest probability, for reasons that will soon become clear.

> Typically, mathematicians' mindsets are biased towards the Platonist system of an ideal, hypothetical and abstract world.

What led to Mr. Hazelton's worries? His narrow mathematical mindset did not take out-of-school mathematics into account. Typically, mathematicians' mindsets are biased towards the pure mathematics system dictated by our curriculum—the Platonist system of an ideal, hypothetical and abstract world. Perhaps Mr. Hazelton, our fictional representative mathematics teacher, needs to learn about out-of-school mathematics, retool his mindset, and consider the positive implications for his classroom teaching. Before doing so, however, we need to better understand Mr. Hazelton's point of view.

In-School Pure Mathematics

Today, Plato's ideas about mathematics directly and indirectly exert enormous influence throughout Western cultures in general, and over mathematics education in particular (Ernest, 2019; Linnebo, 2018). In essence, Plato's philosophy celebrates the purity of the mind while holding worldly matters in contempt. It contends that mathematical ideas existed before humanity existed, and that therefore, these ideas have been *discovered* by mathematicians. This shields mathematics from any doubts expressed about the truth of its conclusions, the ethics of its influence, and the ethics of mathematicians (Ernest, 2018).

From Plato's philosophical axioms, he and his followers logically deduced "truths," which include the beliefs that mathematics is value-free, culture-free, non-ideological, purely objective in its use, generalizable, and universalist—in other words, the only acceptable mathematics system (Ernest, 2016a,b). Notice that the above list of fundamental features has nothing to do with the role of people in mathematics, conveying a message implied in most mathematics textbooks: mathematics is separate from the human condition. These features express "the antithesis of human activity—mechanical, detached, emotionless, value-free, and morally neutral" (Fyhn, Sara Eira, & Sriraman, 2011, p. 186). No wonder learners who have a humanistic outlook on life assess school mathematics as uninteresting or even objectionable.

Plato's ideas have stiff competition from mathematicians and mathematics educators who contend that humans are the source of mathematics. They oppose the narrowness of the age-old (24-century-old, to be exact) Platonist view conventional in school mathematics (e.g., Ernest, 2016a,b; Ernest, Sriraman, & Ernest, 2016; Sriraman, 2017). Take, for instance, the Platonist claim that mathematics is culture-free. Anthropologist Hall (1976) asserted that Plato's concept "purity of mind" (p. 192) and his philosophical assumption that the universe is made up of abstract mathematical objects that mathematicians discover amounts to an intellectual mirage: *"What has been thought of as the mind is actually internalized culture"* (p. 192, original emphasis). In other words, mathematics can be described as a *cultural* enterprise, a position that contradicts the messages conveyed by most mathematics curricula that indirectly promote Plato's philosophy.

> Plato's ideal of pure mathematics is not only cultural, it is value-laden, ideological, and with degrees of subjectivity in its use.

It turns out that Plato's ideal of pure mathematics is not only cultural, it is value-laden, ideological, and with degrees of subjectivity in its use (Ernest et al., 2016; Larvor, 2017). More specifically, Western mathematics is guided by such ideologies as superiority, purism and quantification, and embraces values such as truth, rationalism, universalism, objectivism, and beauty. While we may speak of the *culture of Western mathematics*, Platonists insist that their mathematics is the *only* true mathematics, and that it is culture-free. Consequently, they strongly object to it being called "Western."

Out-of-School Mathematics

A well-known research study by Hoyles, Noss and Pozzi (2001) illustrates out-of-school contextualized mathematics located in the real world, and not in Plato's purity of the mind. Setting aside the ideology[1] of quantification (e.g., the doctrine of letting test scores and their means control humans and their society), they decided to spend a total of 80 hours on the ward with 12 pediatric clinical nurses to observe nurses as they prepared dosages for their patients, watching carefully to see how many mistakes they make. During their visits, the researchers observed 30 instances of drug administration that involved 26 different types of ratio calculations. Their conclusion? Zero mistakes. Mr. Hazelton's positive outcome in the hospital really *was* predictable. Hoyles and colleagues concluded, "Most nursing literature on drug calculation…in mathematics education focused on individual performance on '*decontextualized*' written tests" (p. 11, emphasis added). These pediatric nurses' 30 calculations, however, were all *contextualized* in the reality of a hospital ward.

Obviously, these results (75% to 79% versus 100%) suggest that abstract, decontextualized mathematics (mostly in-school mathematics) must be significantly different from concrete, contextualized mathematics (mostly out-of-school mathematics). The two types of knowledge have also been referred to as "declarative" and "procedural" (Chin et al., 2004).

[1] In general, an ideology is a doctrine that tells people or institutions how to treat others.

Mr. Hazelton's worry, then, suggests that he had a blind spot in his mathematical mindset: While he paid attention to the declarative research results, he did not consider the nurses' procedural mathematics proficiency and therefore underestimated their mathematical ability.

What kind of contextualized proportional reasoning did the clinical nurses actually use? Their professional training provided them with specific procedures to be used in the workplace. Here is one such algorithm (Hoyles et al., 2001):

The actions that a nurse would perform in identifying and handling three quantities when preparing a drug: Look at the drug dose prescribed on the patient's chart ("what you want"); next note the mass of the packaged drug on hand ("what you've got") and then the volume of solution ("what it comes in"). (p. 13)

Selden and Selden (n.d., website quote) translated this on the Mathematical Association of America website as:

$$\frac{Drug\ prescribed}{Dose\ per\ 'measure'} \times Number\ of\ measures$$

Mathematicians' mathematical mindsets may lead them to see the nurses' procedural proportional reasoning from a purely algebraic perspective as:

$$Dosage = \frac{w \cdot v}{g},$$

where w is *w*hat you need, v is *v*olume of solution, and g is what you've *g*ot.

Notice that while the three algorithms (i.e., the nurses', Selden and Selden's, and the algebraic one) are expressed in three different languages, they share a similar meaning. However, they are not the same. Similar to their research with nurses, Hoyles and colleagues (2001) investigated the practices of investment-bank employees and commercial pilots. They discovered a similar disconnect between these employees doing decontextualized (declarative) mathematics and performing contextualized (procedural) mathematics. According to the behaviour of these people who do out-of-school mathematics, they use a different type of knowledge than in-school mathematics people do (i.e., procedural vs declarative, respectively).

Further insight comes from Devlin's (2005) research into the "street mathematics" of school-aged children who work as street market vendors in a third-world country. With permission, the researchers unobtrusively listened to the vendors as they interacted with their customers. The purpose was to assess the accuracy of the children's out-of-school, contextualized, street mathematics. It was found to be consistently accurate more than 98% of the time. Next, the researchers had each street vendor do some typical in-school decontextualized mathematics. The results were as follows:

They averaged only 74% when presented with [textbook-like] market-stall word problems requiring the same arithmetic, and a mere 37% when *virtually the same problems* were presented to them in the form of a straightforward arithmetic test. (website quote, emphasis added)

Context matters! In-school mathematics (i.e., abstract Platonist mathematics devoid of context) is a different kind of knowledge than out-of-school mathematics (i.e., reality-based mathematics immersed in context), for most, but not necessarily all learners.

Did you notice that mathematician Devlin belies a strong Platonist bias in judging the two situations as "virtually the same problems"? From the young market vendor's perspective, however, they are obviously *not* the same. In my view, we have found a blind spot in Devlin's mathematical mindset, similar to Mr. Hazelton's blind spot. Devlin does not seem to understand the vendors' mathematical perspective: in other words, the concept of out-of-school procedural mathematics. Perhaps that is due to the Platonists' insistence that their mathematics is the only true mathematics.

Most elementary teachers do not have such blind spots, moving back and forth from out-of-school to in-school mathematics daily. This is a matter of common sense to the teachers. This is because most elementary teachers know that most of their learners *must* be taught mathematics in context (Hill et al., 2008), where "mathematics becomes best understood by how it is used" (Barta et al., 2014, p. 3). Otherwise, learning rarely occurs. They therefore relate their learners' world (out-of-school mathematics) to the curriculum content, the latter being written in a vocabulary of abstract in-school mathematics, (e.g., quantity, patterns, sorting, shape and space). This learning is aided by visuals, manipulatives, and through play for the youngest learners (Brokofsky, 2017).

> Most learners must be taught mathematics in context, where mathematics becomes best understood by how it is used.

Out-of-school mathematics is typically encountered in three cultural venues: home, community, and employment in most workplaces *not* requiring mathematics specialists highly proficient in algebra, geometry, and calculus. For instance, Martin (2014) and Tencer (2016) identified the following as the highest-paying jobs for people who hate mathematics: business managers, bank clerks, financial advisors, office workers, sales people, many health care workers, therapists, most trade workers, university professors of humanity subjects, lawyers, police officers, many entrepreneurs, and power plant operators, to name a few.

Applied Mathematics

Mathematics teachers often talk about *applying* the curriculum's in-school pure mathematics to out-of-school contexts. But how much does this make sense, now knowing that nurses, financial advisors, commercial pilots, and street vendors, for instance, master concrete procedural mathematics directly on the job? *Applying* abstract pure mathematics, it turns out, is not necessary for them. In fact, it could be risky. Recall how well nurses and street vendors did on their decontextualized in-school pure mathematics tests.

Who *does* apply pure mathematics, then? And how do they do it, exactly? In general, scientists, engineers, mathematics teachers, architects, medical doctors, and some mathematicians apply pure mathematics on the job. In fact, anyone can do it, as long as their worldview or self-identity happens to harmonize with that of a mathematician to a sufficient degree (Nasir, 2002). However, without being familiar with the research, math-oriented people such as our fabled Mr. Hazelton are likely to *assume* that nurses, financial advisors, and pilots also apply pure mathematics at work. Why do they assume this? Here are several potential reasons:

1. They have a mathematical blind spot. They do not sufficiently understand out-of-school mathematics or humanistic self-identities.
2. They subscribe to a Platonist ideology of purism, which favours purity over practicality, and assigns high status to pure abstract mathematics and low status to applied mathematics. And, after 12 years of reading this message conveyed in textbooks and hearing it from some secondary school teachers, most of the general public accepts the ideology of mathematical purism as part of their mainstream Canadian culture, without critically thinking about the ethics of doing so (Ernest, 2016b, 2019).
3. Their worldview enables them to "see" abstract, pure mathematics in the world around them. To them, this lens is just common sense, and they assume that anyone can do it. They are oblivious to their unconscious mental act of projecting pure mathematics concepts onto the world around them, a process that mathematicians call "applying mathematics."
4. They are unfamiliar with the mechanism for applying mathematics consciously or unconsciously, a process that has four steps: recalling, superimposing, comparing, and judging (or three steps: "superimposing, deconstructing, and reconstructing"; Aikenhead, 2017, p. 101). It proceeds as follows:
    a. Their mind invents *images* of mathematical abstractions they have learned (Einstein, 1930; quoted in Director, 2006), which are *recalled* as a result of seeing features in the real world.
    b. They *superimpose* a recalled image onto that feature.
    c. They *compare* their mind's image and the observed feature.
    d. Their mind *judges* the closeness of fit between the two. When there is a close enough fit, the person identifies that feature in terms of the image superimposed on it, which in turn is associated with the original mathematical abstraction. Voilà: The abstraction is perceived in the real world.

When you are conscious of these steps, you have a realistic in-depth understanding about what "applying math" means. Note that step 4.a above, which involves inventing images of mathematical abstractions, can be very challenging for the majority of Saskatchewan high school graduates (as explained below), and largely irrelevant outside of school. Being conscious of this fact helps you avoid Mr. Hazelton's and Professor Delvin's blind spot. At work, rather than applying an abstraction, a person's mathematical procedural knowledge is most often recalled or further refined in order to get a specific job done. Out-of-school mathematics has meaning in relation to a concrete task and, likely, to many personal and idiosyncratic associations. Abstractions, however, are universal by definition. However, they can sometimes only get in the way of completing a task efficiently.

> The prerequisite to applying pure mathematics is to have understood in depth the abstract concept in the first place.

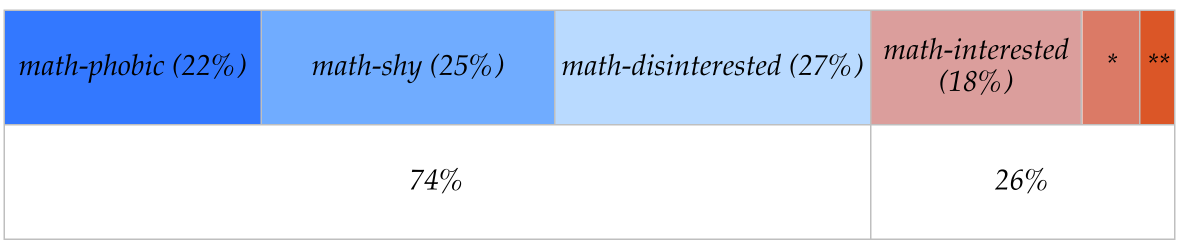
The most important thing to remember about the act of unconsciously applying pure mathematics is that the prerequisite is to have understood in depth the abstract concept in the first place (from which the human mind constructs a representative image; Einstein, 1930). However, this deep understanding is only available to learners to the extent that their worldview or mathematical self-identity harmonizes with a mathematician's (Nasir, 2002).

Mathematical Mindsets: Students *vs* Mathematicians

Learners whose worldviews tend not to harmonize with a mathematician's worldview to various degrees comprise about 70-74% of high school graduates in Saskatchewan, and include future nurses, financial advisors, and pilots (Card & Payne, 2017; Meyer & Aikenhead, 2021a). To varying degrees, these high school graduates tend to be interested in the humanities and tend to have negative predispositions to pure mathematics. Indeed, an AP-AOL News poll conducted by Ipsos (2005) found that about 37% of young adults "hate math" (p. 2). According to the same poll, only 23% of adult respondents stated that mathematics was their favourite subject, while 51% chose a humanities subject as their favourite. On the surface, this polling result does not make secondary school mathematics look good. However, we need to look deeper.

Imagine a spectrum of learner diversity. At one extreme are learners whose worldviews and self-identities (Nasir, 2002) generally harmonize with that of their mathematics teacher (the right-hand side of Figure 1). At the other extreme are learners who often develop serious psychological or physiological anxieties when forced to think mathematically, especially when being assessed (Ernest, 2018; Maloney, Fugelsang & Ansari, 2016). Figure 1 is structured in line with PISA's six student proficiency levels (OECD, 2019) to avoid simplistic dichotomous reasoning.

Figure 1. Distribution of Saskatchewan Grade 12 learners' degrees of harmony between their self-identities and mathematicians' worldviews. Not for streaming learners. The percentage proportions of learners in each category arise mostly from PISA 2018 proficiency data, bounded by other research sources. Source: Meyer and Aikenhead (2021a).



* *math-curious* (5%)
** *math-oriented* (3%)

These six categories must be treated as being very flexible and tentative, because learners' designation to a category depends on many changeable factors (e.g., the teacher, topic, grade level, classroom environment, degree of past success, season, etc.). The categories have been proposed for the purpose of discussing learner diversity, and certainly not for streaming purposes. Fine distinctions between them are not made in this article.

Figure 1 shows the percentage of learners likely to be in each of the six categories, primarily based on PISA 2018 data (OECD, 2019) and modified by Meyer and Aikenhead (2021a) to conform with data published in the research literature (Card & Payne, 2017; Frederick, 1991). The data create a skewed distribution of learners in favour of those who would avoid high school mathematics if they could. This spectrum of learner diversity produces several notable issues to consider.

Even though the math-phobic, math-shy, and math-disinterested learners generally do not perform well on mathematics tests, they can excel at doing a small number of specific kinds of mathematical tasks (Louie, 2017). These could be cultivated by their teacher, of course. In line with Boaler's (2015) "growth mindset" discussions, learners can move towards the right side of Figure 1 (Boaler & Confer, 2017), even if not far enough to "learn math to the highest levels" (youcubed at Stanford University, 2020, p. 1). My experience tells me, however, that many teachers seem to believe this youcubed message on its commercial website. To me, this expectation seems an unfair burden for teachers to bear. Still, teachers should strive to understand and empathize with the math-phobic, math-shy, and math-disinterested learners and learn how to communicate more effectively with these learners so as to help them develop a more positive mathematical mindset.

> Teachers should strive to understand, empathize with, and communicate more effectively with the math-phobic, math-shy, and math-disinterested learners.

One of the goals of the Saskatchewan curriculum is "understanding mathematics as a human endeavour" (Saskatchewan Ministry of Education, 2008, p. 9). This goal opened the gate, so to speak, to include Indigenous ways of knowing in a recently constructed provincial mathematics pedagogical resource for teachers and leaders of mathematics in Saskatchewan entitled *SaskMATH* (Provincial Education Sector, 2021, website quote). It is legitimate, therefore, to treat Indigenous mathematizing as out-of-school mathematics content with which to enhance your mathematics classes and reach even your most math-phobic learners. You will be surprised to discover how many humanistic contexts there are in which students can engage with mathematics in their real world (Aikenhead, 2021a,b).

Indigenous Mathematizing and Perspectives

Out-of-school mathematics is related to Indigenous mathematizing because each is preoccupied with procedural knowledge. In other words, they share a focus on getting a job done. However, they differ in the type of understanding that results: intellectual versus wise, as explained later in this section. The term "mathematizing" is used because Indigenous languages are "verb-based" (Aikenhead, 2017, p. 87), whereas Western languages are noun-based. This cultural difference is respectfully acknowledged and accurately described by the verb "mathematizing," in keeping with an anthropological definition of mathematics for any culture: counting, measuring, locating, designing, playing, or explaining quantitatively (Bishop, 1988, pp. 147-151).

Traditional Indigenous mathematics systems continue today, expressed through actions. A McDowell Foundation research report (Duchscherer et al., 2019) describes Grade 5-12 students at Carrot River engaged in the following mathematizing activities, each associated metaphorically with curricular content (shown in brackets along with the name of the lesson creator):

1. pow-wow dancing and beading (polygons; Serena Palmer);
2. playing games (probability and combinatory logic; Danielle Vankoughnett; and two-dimensional spatial reasoning using problem solving strategies; Kevin Duchscherer);
3. hand drumming (multiplication; Serena Palmer);
4. looming (probability trees; Krysta Shemrock);
5. berry picking (number line; Danielle Vankoughnett);

Project leader and knowledge holder, Sharon Meyer, produced three videos that exemplify Indigenous mathematizing. Two are excerpts from exemplary Indigenous culture-based mathematics lessons she taught, which involve birch-bark biting (angles and symmetries) and dream catcher construction (polygons and two-eyed seeing), respectively.

The project report identifies various kinds of supports that teachers need in order to collaborate to produce high quality, Indigenous mathematizing lessons that connects with specific curriculum entries. The report also identifies what teachers need to learn if they have not already, as well as what they may need to *unlearn* in order to facilitate these lessons in a manner that is faithful to their intent.

Conclusion

Indigenous mathematizing shares some values with Western mathematics (e.g., truth, beauty, and the human desire to make sense of the world). However, the two mathematical systems rely upon completely different ways of knowing. While understanding Western mathematics is accomplished according to the *intellectual* tradition of understanding, which is mainly linear, reductionist, analytical, and resides in the brain, understanding Indigenous mathematizing is accomplished through the *wisdom* tradition of understanding—a holistic balance engaging the intellectual, emotional, physical, and spiritual dimensions of the medicine wheel (i.e., the brain, heart, body, and soul).

A concept of importance to Indigenous culture-based school mathematics is *two-eyed seeing* (Hatcher et al., 2009). It means to learn the best from each system, and then use either one, or a hybrid of both. In doing so, we must take care to maintain the cultural authenticity of each system (Garroutte, 1999), lest the Platonist ideology of superiority infringes on the equity spirit of two-eyed seeing, as witnessed during European colonization (Bishop, 1990). However, the effort is worthwhile, as inviting Indigenous mathematizing into the classroom may help math-phobic and math-shy learners give greater attention to relevant curriculum content (Meyer & Aikenhead, 2021b) and thereby move toward a more positive mathematical mindset.

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Glen Aikenhead is Professor Emeritus (2006), University of Saskatchewan, Canada. He earned an Honours BSc (University of Calgary, 1965), Masters of Arts in Teaching (Harvard University, 1966), and a Doctorate in Science Education (Harvard University, 1972). His projects in science and mathematics education over the years have emphasized relevance for learners. Since the early 1990s, Dr. Aikenhead has engaged in cross-cultural school science for both Indigenous and non-Indigenous students. In 2000, he collaboratively developed a community-based series of units entitled Rekindling Traditions. *Since then, he helped edit the Grades 3-9 Saskatchewan science textbook series, which highlights Indigenous knowledge; in 2011, he co-authored* Bridging Cultures, *a resource for teachers implementing science curricula that recognize Indigenous knowledge as a foundational way to understand the physical world; and, in 2014, he published a collaborative, action-research, professional development program for teachers entitled* Enhancing School Science with Indigenous Knowledge. *In 2020, Saskatchewan's science program was recognized by the Conference Board of Canada as leading Canada in reconciliation. Currently, Dr. Aikenhead is involved in a similar innovation for Saskatchewan's mathematics program. He published a monograph,* Enhancing School Mathematics Culturally, *as a 2017 issue in an academic journal. During the 2018-19 school year, Dr. Aikenhead and Plains Cree Knowledge Holder, Sharon Meyer, received a McDowell Foundation grant (Culture-Based School Mathematics for Reconciliation and Professional Development) to collaborate with four Carrot River mathematics teachers to develop Indigenous mathematizing lessons that connected with the curriculum. In 2020, he helped develop the SaskMATH project.*

Intersections

In this column, you'll find information about upcoming math education-related workshops, conferences, and other events. Some events fill up fast, so don't delay signing up! For more information about a particular event or to register, follow the link provided below the description. If you know about an upcoming event that should be on our list, please contact us at thevariable@smts.ca.

Upcoming

Tweaking Number Talks
October 17, 2021
Online, presented by the SMTS

Participants will be introduced and engage in Number Talks with a twist that will give you greater insight into your students number sense. You will see how a small change allows students to be successful, share what they know and produce a product for teachers to use in their assessment and reporting.

Register at www.smts.ca/professional-development/

MCATA Fall Conference
October 22-23, 2021
Online, presented by the Math Council of the Alberta Teachers' Association

The MCATA 2021 Fall Conference is online and free! Featured speakers include Peter Liljedahl, Marian Small, Francis Su, Florence Glanfield, and many more.

Register at bit.ly/MCATA2021Conference

Working With Constraints
October 26, 2021
Online, presented by the SMTS

Sometimes less is more. In this session, we will engage in simple tasks that are filled with rich mathematical opportunities that can be used in your classroom tomorrow.

Participants will leave this session with tasks, resources, and ideas to promote flexible thinking in mathematics.

Register at www.smts.ca/professional-development/

Ongoing

Education Week Webinars
A collection of free and premium virtual broadcasts, including upcoming and on-demand webinars. These virtual broadcasts cover teaching and learning and include webinars on differentiated instruction and the common-core standards. All webinars are accessible for a limited time after the original live streaming date. For all webinars broadcast by Education Week after August 1, 2019, Certificates of Completion are available to all registered live attendees who attend 46 minutes or more of any webinar.

Available at www.edweek.org/ew/marketplace/webinars/webinars.html

Global Math Department Webinar Conferences
The Global Math Department is a group of math teachers that organizes weekly webinars and a weekly newsletter to let people know about the great stuff happening in the math-Twitter-blogosphere and in other places. Webinar Conferences are presented every Tuesday evening at 9 pm Eastern. In addition to watching the weekly live stream, you can check the topic of next week's conference and watch any recording from the archive.

Available at www.bigmarker.com/communities/GlobalMathDept/conferences

NCTM E-Seminars and Webcasts
Presented by the National Council for Mathematics Teachers

E-seminars are recorded professional development webinars with facilitator guide and handouts. E-seminars are free for NCTM members. Webcasts of Annual Meeting Keynote Sessions offer notable and thought provoking leaders in math education and related fields as they inspire attendees at NCTM Conferences.

Available at www.nctm.org/NCTM/templates/ektron/two-column-right.aspx?pageid=75420

Tangents

Extracurricular Opportunities for Students

This column highlights local and national extracurricular opportunities for K-12 students interested in mathematics, including collaborative, individual, online, and in-person challenges, contests, and camps. For dates, registration procedures and applications, and other information about the contests listed, please head to the contest websites, included below the descriptions. If we have missed an event that should be on our list, please contact us at thevariable@smts.ca.

If you are looking for contests available in other provinces, head to the Canadian Mathematical Society website (cms.math.ca/Competitions/othercanadian).

Contests

Canadian Computing Competition
Held online in February

The Canadian Computing Competition (CCC) is a fun challenge for secondary school students with an interest in programming. It is an opportunity for students to test their ability in designing, understanding and implementing algorithms. Students are encouraged to prepare; see suggestions on contest website. The contest is held online in schools.

More information at https://www.cemc.uwaterloo.ca/contests/computing.html

Canadian Team Mathematics Contest
April 7, 2022
Presented by the Centre for Education in Mathematics and Computing (University of Waterloo)

The Canadian Team Mathematics Contest (CTMC) is a fun and challenging competition for teams of 6 secondary school students in any combination of grades. One teacher and groups of six students participate at their own school over 3 consecutive hours. At least half of the problems in the Individual Event, and many problems in the Team Event, are accessible to students in grade 9 or 10. Junior students will be able to make significant contributions but teams without any senior students may have difficulty completing all the problems. Teams from Canadian schools can register for a lottery to participate in-person or virtually in an event hosted by the CEMC, or may participate unofficially in their school on any day after the official contest date.

More information at www.cemc.uwaterloo.ca/contests/ctmc.html

Caribou Mathematics Competition
Held six times throughout the school year

The Caribou Mathematics Competition is a worldwide online contest that is held six times throughout the school year. Each of these days, five contests are offered, one for each of the grade levels 3/4, 5/6, 7/8, 9/10 and 11/12. Contest questions and solutions are now offered in English, French, Persian, Mandarin, Ukrainian, Azerbaijani, Khmer, and the list keeps growing! All participants receive certificates for participation and achievement, and top-scoring students can earn cash prizes.

Caribou contests differ from other contests in that they:

- are offered six times per school year (October - May),
- are designed to be accessible and enjoyable to students of all levels, not just those who excel in mathematics,
- cater to students from grade 2 all the way through to grade 12,
- contain interactive questions loved by students,
- feature mathematical puzzles rather than strictly knowledge-based questions,
- require minimal time and effort from schools and teachers since contests are held online and graded automatically,
- come with results and statistics available on the evening after the contest,
- provide 250 video solutions to selected questions,
- offer interactive practice access to contests from previous years and detailed written solutions

More information at cariboutests.com

Euclid Mathematics Contest
Written in April
Presented by the Centre for Education in Mathematics and Computing (University of Waterloo)

The Euclid Mathematics Contest is an opportunity for students in their final year of secondary school and motivated students in lower grades to have fun and to develop their mathematical problem solving ability. Most of the problems are based on the mathematical curriculum up to and including the final year of secondary school. Some content might require students to extend their knowledge and the best way to familiarize oneself with commonly appearing topics is to practice using past contests. The contest is written by individuals in schools.

More information at www.cemc.uwaterloo.ca/contests/euclid.html

Fryer, Galois, and Hypatia Mathematics Contests
Written in April
Presented by the Centre for Education in Mathematics and Computing (University of Waterloo)

The Fryer, Galois and Hypatia Math Contests are an opportunity for students to write a full-solution contest. They are fun way to develop mathematical problem solving skills through a written mathematical activity. For students in Grades 9 (Fryer), 10 (Galois) and

11 (Hypatia). Questions are based on curriculum common to all Canadian provinces. Rather than testing content, most of the contest problems test logical thinking and mathematical problem solving. The contest is written by individuals in schools.

More information at www.cemc.uwaterloo.ca/contests/fgh.html

Gauss Mathematics Contests
Written in May
Presented by the Centre for Education in Mathematics and Computing (University of Waterloo)

The Gauss Contests are an opportunity for students in Grades 7 and 8, and interested students from lower grades, to have fun and to develop their mathematical problem solving ability. Questions are based on curriculum common to all Canadian provinces. The Grade 7 contest and Grade 8 contest is written by individuals and may be organized and run by an individual school, by a secondary school for feeder schools, or on a board-wide basis.

More information at www.cemc.uwaterloo.ca/contests/gauss.html

Opti-Math
Written in March
Presented by the Groupe des responsables en mathématique au secondaire

A French-language mathematics challenge for secondary students who would like to exercise and develop their problem-solving skills.

Les Concours Opti-Math et Opti-Math + sont des Concours nationaux de mathématique qui s'adressent à tous les élèves du niveau secondaire (12 à 18 ans) provenant des écoles du Québec et du Canada francophone. Ils visent à encourager la pratique de la résolution de problèmes dans un esprit ludique et à démystifier, auprès des jeunes, les modes de pensée qui caractérisent la mathématique. Le principal objectif des Concours est de favoriser la participation bien avant la performance. La devise n'est pas : « que le meilleur gagne » mais bien « que le plus grand nombre participe et s'améliore en résolution de problèmes ».

More information at www.optimath.ca/index.html

Pascal, Cayley, and Fermat Contests
Written in February
Presented by the Centre for Education in Mathematics and Computing (University of Waterloo)

The Pascal, Cayley and Fermat Contests are an opportunity for students in Grades 9 (Fryer), 10 (Galois)m and 11 (Hypatia) to have fun and to develop their mathematical problem solving ability. Early questions require only concepts found in the curriculum common to all provinces. The last few questions are designed to test ingenuity and insight. Rather than testing content, most of the contest problems test logical thinking and mathematical problem solving. The contest is written by individuals in schools.

More information at www.cemc.uwaterloo.ca/contests/pcf.html

The Virtual Mathematical Marathon
Supported by the Canadian National Science and Engineering Research Council

The virtual Mathematical Marathon has been developed by an international team of mathematicians, mathematics educators, and computer science specialists with the help of the Canadian National Science and Engineering Research Council.

The main activity is a competition allowing students to enjoy solving challenging mathematical problems all year around. Students can join the game at any time and at no cost, simply creating an individual profile with an individual username and a password. Available in French.

More information at www8.umoncton.ca/umcm-mmv/index.php

Yang Math League
Levels: Grade 8 and under; Grades 9 to 12
Frequency: Weekly
Time: 30 minutes at any convenient time on Saturday or Sunday
Topics: Full range of school mathematics

The Yang Math League (YML) is entirely organized and run by Saskatchewan's own Stephen Yang, a talented and passionate Grade 10 math student in Saskatoon. Students receive the six weekly questions through email each Saturday morning at 9 am and can choose when they do them that weekend. They submit their answers on a Google form that is scored automatically, and receive their scores back on Monday evening along with their cumulative score and the names of the perfect scorers. When over 20% of the students ask for a solution to a question, Stephen posts a YouTube video within a week.

Students can participate for as many or few weeks as they want and take a break for one or several weeks. Students who have participated consistently see a growth in their ability to solve tough mathematical problems.

To register, use the following link: https://bit.ly/2KpRAmX
Also check out Stephen's YouTube channel, which includes solutions to a variety of tough math questions from contests: https://bit.ly/39E2C0t

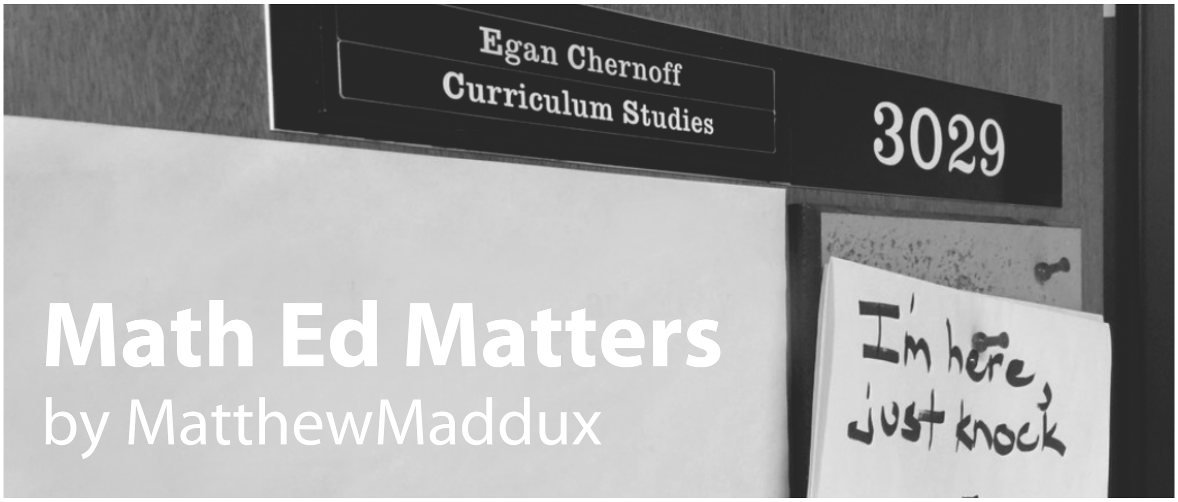
Resources

Canada Math YouTube Channel
CanadaMath is a large online collection of tutorial videos for North American Grades 7-12 mathematics competitions, including the Gauss, Pascal, Cayley, Fermat, and Euclid contests.

Access the videos at https://www.youtube.com/CanadaMath

Canadian Mathematical Society Resources
Practicing problems is a key element to prepare for math competitions. The CMS offers a variety of resources for students who are looking to build their problem solving skills and succeed in competitions.

See https://cms.math.ca/competitions/problem-solving-resources/



Math Ed Matters by MatthewMaddux *is a column telling slightly bent, untold, true stories of mathematics teaching and learning.*

Minding the Generation Gap: One Step Forward or Two Steps Back?

Egan J Chernoff
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I love the phrase "Ok, boomer." For those not yet in the know, "Ok, boomer" is a dismissive retort used by Generation Z to mock members of the Baby boom generation, i.e., the Baby boomers. Say, for example, a bespectacled, white-haired, high-waisted-pant-wearing, elderly man mutters to his friend, "Damn kids these days, can't even make change without a calculator" as they pass the food court again on Lap 2 of their mall-walking "adventure." Any member of a younger generation within ear shot could then repudiate said comment without even having to look up from the phone super-glued to their hand by simply countering with a dismissive, derisive, "Ok, boomer." Shots fired. As I said, I love it.

As a member of Generation X, I have had my fair share of interactions with members from the Baby boom generation. For the record, and I'm speaking here to both Millenials and Gen Z, the "horror stories" that you've heard are absolutely true. For example, back in my day, there was a simple and guiding principle when it came to learning mathematics: "In mathematics, you never understand things; you just get used to them." That's right, no colorful ones, tens, hundreds, and thousands blocks on my desk when I was learning place value. No colorful Cuisenaire rods when it was time to learn fractions. No Teddy Bear counters, no Unifix cubes, and definitely no Geoboards. No pattern blocks strewn about my desk whilst getting applauded by my teacher for a picture that I made while I was supposed to be doing something else entirely. Now, did I completely understand everything that was going on when it came to learning, for example, fractions? No. What I did learn, though, and I'm

> Back in my day, there was a simple and guiding principle when it came to learning mathematics: "In mathematics, you never understand things; you just get used to them."

paraphrasing von Neumann here, was to "get used to it" and, more importantly, to "get on with it." And that's just what I did.

"But that's not sustainable," I can hear members of the younger generations saying at this point. Well, yes and no. Speaking for myself, I was able to kick the can way, way down the road. "Getting used to things" got me through elementary school math, middle school math, high school math, and even the first two years of a mathematics major at university. But yes, eventually, at some point we must all meet our Maker. And I remember meeting my Mathematics Maker like it was yesterday.

Moving into the third year of my mathematics major, having taken practically every calculus course that was on the books, it was time to branch out. All of a sudden, in one fall semester, I was taking Statistics, Elementary Number Theory, Euclidean Geometry, and other math courses. After the first week of lectures for the Euclidean Geometry course, we were handed our first assignment. Looking over the questions, things didn't look too bad. I slid the assignment into my binder and, instead of heading to the Math Help Centre, which is what all the other math majors did, I strolled across campus to attend my geography class. Almost everybody was gone by the time I got to the Math Help Centre that day; after all, it was early in the semester. Good—this meant that I could focus on my homework, get it done, and turn my attention to what I had planned for the weekend. I pulled out the assignment, read the first question, which read something like, "Prove that Angle-Side-Angle proves congruence." And there it was, in all its glory: irrefutable proof that I was way out over my skis.

I was a deer in the headlights, but doing my best to not project this to the other students or tutors in the room. "Something must be wrong here," I thought to myself. I looked down at the other questions on the assignment. The next questions: "Prove that Side-Angle-Side proves congruence," and "Prove that Angle-Angle-Side proves congruence." To be honest, I didn't even understand the question. "I don't get it," I thought to myself, "Angle-Side-Angle already proves congruence. So do Side-Angle-Side and Angle-Angle-Side. What is there to prove?" Getting nowhere, I finally swallowed my pride and consulted a trusted confidant who, it turned out, took the same high school math classes that I did. "Think back," he said, "He just *told* us all that Angle-Side-Angle proves congruence. Right? Well, now we have to *prove* that ASA proves congruence." And that's when everything, and I mean everything, about my school mathematics career snapped into focus.

"What is there to prove?" And that's when everything, and I mean everything, about my school mathematics career snapped into focus.

Of course, this experience is not unique. Famously, there's the end scene of *The Usual Suspects* (Spoiler alert!!), told through flashbacks, where agent Dave Kujan realizes who Keyser Soze is, albeit a little too late. In my own life, my life as a math student flashed before my eyes as I sat there trying to wrap my head around actually proving that something proves congruence. I realized that I never *really* understood why, for example, those parabolas were moving left and right in Grade 11. Rather, I had just memorized a rule that involved seeing either a plus or minus and moving left or right accordingly. I recalled another time when I was able to answer questions, but not explain what was going on to a classmate who asked for help with his calculus homework, realizing that this stemmed from not having ever truly understood the fundamental theorem of calculus. Other examples kept coming to mind, but I had to put an end to the flashbacks and focus. There was work to do, and I had no idea how to do it.

Having made it so far in mathematics on such a weak foundation was not necessarily the norm, even in those days. Somehow, I had slipped through the earlier weeding-out process. “'Weeding out? They can't do that!” I hear members of the younger generations exclaim. They could, and they did! It went like this... On one of the first few days of your university experience, you would show up to your very first calculus class, along with hundreds of other people, where, unbeknownst to you, you had to take a pop quiz. Should you fail this quiz, as some poor first-years would soon realize, you would be politely asked, or in some cases not-so-politely told, to drop the class. At that point, though, my house of cards was still standing tall. I passed the quiz and continued to live in blissful ignorance of my own ignorance for another two years, until that fateful day in the Math Help Centre.

> The day-one calculus quiz is a Rorschach Test for how different generations think about the teaching and learning of mathematics.

This day-one calculus quiz, I contend, is a Rorschach Test for how different generations think about the teaching and learning of mathematics. Let me explain. As traumatic as the experience may be, members of older generations are apt to see the quiz as actually *protecting* first-year calculus students, reasoning that it's better for a student to know that they had little to no chance at passing a course on day one, as opposed to investing a considerable amount of time, effort, and money only to fail the course. The quiz, then, was akin to looking into a crystal ball, seeing the future, knowing the outcome months in advance, and sparing all of those for which it didn't look good. As Mike Tyson famously said, “Everyone has a plan until you get punched in the mouth.” Mathematically speaking, then, the day-one calculus quiz was given out to see if you could survive getting punched in the mouth.

Members of younger generations likely view the this exam differently. Rather than focusing on the mathematical content of the exam, they are concerned about its affective consequences. Is accurate assessment of learning even possible, or equitable on the first day of class? Should the students at least receive some advance notification? The lack of individualized exam accommodations, as well as the prohibition on calculators, would spur further discussions about justice and equity. Members of older generations, particularly those who endured and made it past the first-day calculus exam, are likely see these issues as inane; “Snowflakes!”, retort the most hard-nosed among them. Alas, there is no substitute for experience. I guess, then, it's time for members of older generations to start considering what life for mathematics teachers and learners will look like if you never, ever get punched in the mouth.

Of course, it’s no surprise that members of different generations are prone to interpreting the same phenomenon in very different ways. Just recently, I was having another version of the “Do you see a duck or a rabbit?” discussion with a colleague of mine who works in the math department at our university. They, a fellow member of Generation X, shared some rather illuminating details stemming from their recent teaching of the Mathematics for Early and Middle Years Teachers course. On the one hand, they noted, the majority of students still struggled with basic fraction arithmetic and some even with their times tables. On the other hand, they noticed a marked improvement in appreciation for and attitude towards mathematics. So, do you see a duck or rabbit? Whether that's one step forward or two steps back, I guess, is a matter of perspective.

No matter the generation you belong to, the world is changing for all of us. Me, I'm still processing the fact that some students are still learning their times tables during their first year of university. I'm also processing the pride with which future math teachers declare to me on the first day of class, “I don't read,” after asking them to introduce themselves and share their favourite book, movie, TV show, etc. That's right, we do introductions and a pop quiz on the first day of my class—I'm not a monster.

Maybe it's because I'm currently stuck between a few generations that I can see both sides of the coin, but that won't always be the case. Twenty-five years from now (which, if everything goes according to plan, is when I'll retire), I'll be the fodder for the updated version of “Ok, boomer.” Bring it on, I say. I can't wait. I predict the younger generation will make some sort of dismissive play on the “X” of Generation X. But don't expect that play on “X” to be mathematical. Like what happened to learning Latin, there is no way we will be teaching and learning mathematics in school in the future. I do, however, look forward to being proven wrong.

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Saskatchewan Mathematics Teachers' Society