The Variable

Presented by the Saskatchewan Mathematics Teachers' Society

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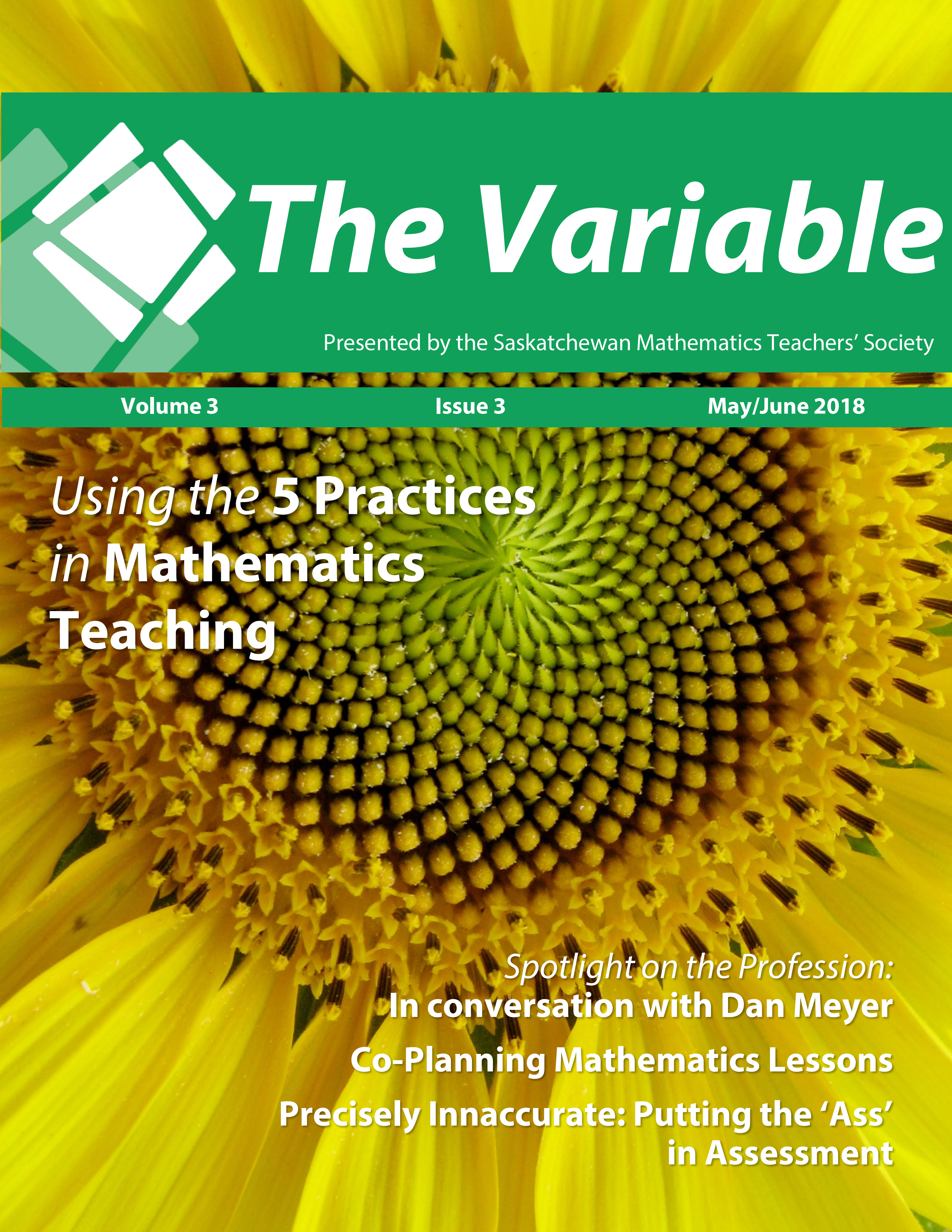


Using the **5 Practices** *in* **Mathematics Teaching**

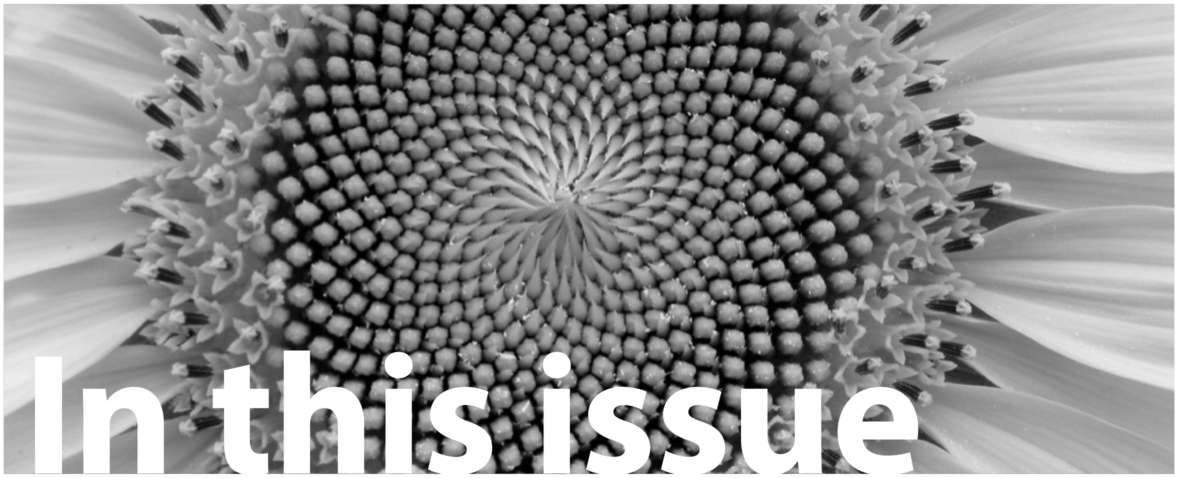
Spotlight on the Profession:
In conversation with Dan Meyer

Co-Planning Mathematics Lessons

Precisely Innaccurate: Putting the 'Ass' in Assessment

Saskatchewan Mathematics Teachers' Society

AFFILIATE
NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS



Cover Image

Mathematical biologists love sunflowers. The flowers are one of the most obvious demonstrations of the Fibonacci sequence, a set in which each number is the sum of the previous two (1, 1, 2, 3, 5, 8, 13, 21…). In this case, the telltale sign is the number of different seed spirals on the sunflower's face. Count the clockwise and counterclockwise spirals that reach the outer edge, and you'll usually find a pair of numbers from the sequence: 34 and 55, or 55 and 89, or—with very large sunflowers—89 and 144. *Read more at* www.sciencemag.org/news/2016/05/sunflowers-show-complex-fibonacci-sequences

Image source: Postpischl, L. (2007). SunFlower: The Fibonacci sequence, golden section [Digital image]. Retrieved from https://www.flickr.com/photos/lucapost/694780262

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Notice to Contributors

The Variable welcomes a variety of submissions for consideration from all members of the mathematics education community in Canada and beyond, including classroom teachers, consultants, teacher educators, researchers, and students of all ages, although we encourage Saskatchewan teachers of mathematics as our main contributors. Submitted material is assessed for interest, presentation, and suitability for the target audience.

Please submit articles by email to thevariable@smts.ca in Microsoft Word format. Articles should be limited to 3000 words or less; authors should also include a photo and a short biographical statement of 75 words or less. Accepted contributions are subject to revision. Editors reserve the right to edit manuscripts for clarity, brevity, grammar, and style.

Saskatchewan
Mathematics
Teachers'
Society

The Saskatchewan Mathematics Teachers' Society presents ...

#SUM2018

November 2-3, 2018

Who:	K-12 teachers, coaches, consultants, coordinators, superintendents and directors
Where:	Circle Drive Alliance Church
When:	November 2-3, 2018
Cost:	$165 (early registration) | $210 (regular)

Keynote Presenters

Lisa Lunney Borden
St. Francis Xavier University

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Call for Contributions

***The Variable* is looking for contributions from all members of the mathematics education community**, including classroom teachers, consultants, and teacher educators. Consider sharing a favorite lesson, an essay, a book review, or any other work that may be of interest to mathematics teachers in Saskatchewan. If accepted for publication, your article will be shared with a wide audience of mathematics educators in Saskatchewan and beyond.

We also welcome student contributions in the form of artwork, stories, interesting problem solutions, or articles. This is a great opportunity for students to share their work with an audience beyond that of their classroom, and for teachers to recognize students' efforts during their journey of learning mathematics.

All work is published under a Creative Commons license. **To submit or propose an article, please contact us at thevariable@smts.ca**. We look forward to hearing from you!

Ilona & Nat,
Editors

Message from the President

Happy May! While this month is the homestretch in classroom land, it's the time when the Saskatchewan Mathematics Teachers' Society and the Saskatchewan Professional Development Unit (my wonderful day job) begin making plans for next year. I absolutely love the process of reviewing feedback and evidence from the year at hand and laying down a path for the new year. There is much to celebrate from 2017-18 in terms of the work of the SMTS, and I'm looking forward to seeing how we continue to grow in the new year.

Although the Saskatchewan Understands Math (SUM) Conference is an annual celebration, our partnership in 2017 with SPDU and the Saskatchewan Educational Leadership Unit made it extra special. We certainly hope it was an experience that has made you mark your calendar for SUM 2018 (November 2-3). If you haven't yet heard, our exciting news is that Lisa Lunney-Borden will be joining us as keynote speaker in 2018. The are also a few other new plans for SUM, but we're not ready to spill quite yet. Stay tuned!

SUM Conference also held our Annual General Meeting, which had us saying goodbye to some veteran executive members as they move onto new things and saying hello to some new directors. It's exciting to think of the ideas and energy our new members will bring to the group.

We're also saying happy second birthday to *The Variable*! This year has seen such an amazing display of talent and generosity on behalf of everyone who contributes. We grew our editing capacity with the addition of a co-editor, and we are continuing to expand our range of content and contributors.

Lastly, we're celebrating a year full of being involved in the larger picture of the teaching and learning of mathematics in Saskatchewan. While we're always sad about bad math press, we are excited to be a voice for the teachers of Saskatchewan at the provincial level. We're advocating for you, our members, in the media, with the Ministry, with the Saskatchewan Teachers' Federation, and, soon, with the National Council of Teachers of Mathematics.

Which bring me to ask, as always - what else would you like to see from us? As you look forward to the 2018-19 school year, what do you need from the SMTS? Reach out to any executive member or to one of our social media accounts with your thoughts. Now is the time to put things in motion. Interested in rolling up your sleeves up to do some work? We still have room for some members at large on the executive! And with that, I wish you a wonderful end of the year, and will see you back here for the summer edition.

Michelle Naidu

Problems *to* Ponder

Welcome to this month's edition of Problems to Ponder! Have an interesting solution? Send it to thevariable@smts.ca for publication in a future issue of The Variable!

Primary Tasks (Kindergarten-Intermediate)

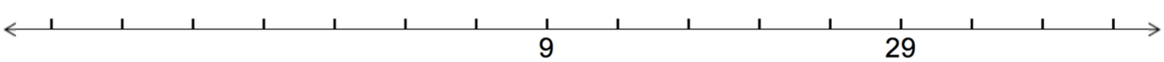
Snap It![1]
In this activity, students make different combinations for a given number.

Materials
10 or more snap cubes per student

Task Instructions
Each student makes a train of connecting cubes of a specified number. On the signal "Snap!", the students break their trains into two parts and hold one hand behind their back.

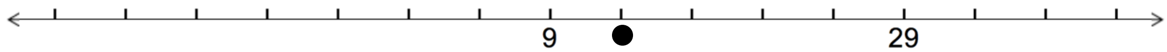
In partners, or in a circle, students show one another their remaining cubes. The other students work out the number of cubes hiding behind their back.

Number Lines in Disguise[2]
This number line is missing some numbers.



[1] Snap it. (n.d.). Retrieved from the YouCubed website at www.youcubed.org/tasks/snap-it/
[2] NRICH. (n.d.). Number lines in disguise. Retrieved from nrich.maths.org/13452

What number should be where the dot is? How do you know?



Where would 0 be on the line?

Damult Dice[3]
In this game, students roll dice and try to reach a target number by adding and multiplying the numbers rolled.

Materials

- 3 dice
- Paper and pencils, or white board and markers for recording work

Task Instructions
Each player takes turns rolling the three dice.

On their turn, a player rolls two of the dice and adds the two numbers. Then, they roll the third die and multiply the sum by the number rolled. This player's score for their turn is the sum of this product plus their previous score.

The first player to reach 200 (or 500, etc.) wins. Prior to the start of the game, players can decide whether the first person to exceed the target number wins, or whether the target number must be reached exactly. In the latter case, a player who cannot make a product that will not exceed the target number must skip their turn.

Adaptations: Play with dice numbered from 1 to 3 and make the target number 100. Give partners a hundreds chart and two tokens to keep track of their score.

Intermediate and Secondary Tasks (Intermediate-Grade 12)

Triangles, Circles, Squares[4]

a) The large triangle (*see next page*) is equilateral. The red triangle is constructed using the midpoints of two sides, and a vertex, of the equilateral triangle.

   What fraction of the whole triangle is the red triangle?

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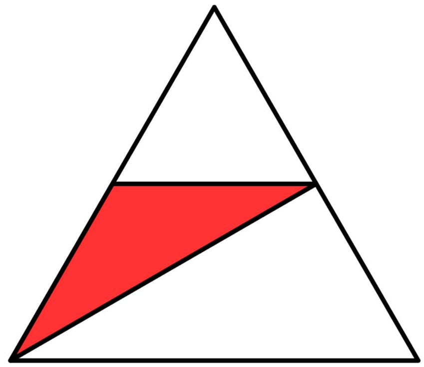
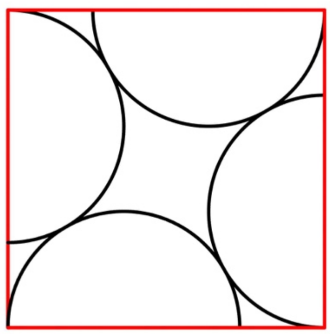
[3] *Adapted from* Finkel, D. (2010, October 1). A game to end all times tables drills: Damult Dice [Blog post]. Retrieved from mathforlove.com/2010/10/a-game-to-end-all-times-tables-drills-damult-dice

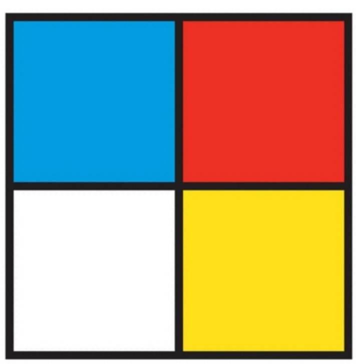
[4] Southall, E., & Pantaloni, V. (2017). *Geometry snacks*. St. Albans, United Kingdom: Tarquin. Images retrieved from www.theguardian.com/science/series/alex-bellos-monday-puzzle



b) Four identical semicircles with radius 2 are constructed in the red square below. What is the area of the square?



Squares Squared[5]
I have four squares. One of them is painted red, one white, one blue, and one yellow, but otherwise they are indistinguishable. I wish to assemble them into one large square. How many ways can I assemble the square?



Now assume that the squares are colored on both sides so that the assembled square can be turned over. In how many ways can I assemble the square?

[5] *Adapted by* Eli Brettler (www.math.yorku.ca/~brettler) *from* Mason, J., Burton, L, & Stacey, K. (1985). *Thinking mathematically*. Essex, England: Prentice Hall.

Extend: Suppose that I now have eight cubes. Two of them are painted red, two white, two blue, and two yellow, but otherwise they are indistinguishable. I wish to assemble them into one large cube with each colour appearing on each face. In how many different ways can I assemble the cube?

Fred and Francine[6]

Fred and Francine are on a run from A to B. Fred runs half the way and walks the other half. Francine runs for half the time and walks for the other half. They both run and walk at the same speeds. Who finishes first?

Extend: Frank joins them and teaches them to jog. Fred now runs one-third of the way, jogs one-third of the way and walks the rest, while Francine jogs for one third of the time, runs for one third of the time, and walks the rest.

Who finishes first? Has Frank helped them to finish sooner or later than previously?

Have a great problem to share?

Contribute to this column!
Contact us at thevariable@smts.ca.
Published problems will be credited.

[6] *Adapted from* Mason, J., Burton, L, & Stacey, K. (1985). Thinking mathematically. Essex, England: Prentice Hall.

Alternate Angles

Alternate Angles is a bimonthly column on problems from multiple perspectives: various methods that could be used to solve them, insights we get from their solution, the new paths that they can lead us to once they have been solved, and how they can be used in the classroom.

Funny Fractions

Shawn Godin

Welcome back, problem solvers. In the last issue, I left you with the following problem:

> One day, Alice encountered the number $4\frac{4}{3}$. She went to simplify it, but mistakenly thought that the expression meant $4 \times \frac{4}{3}$. When she showed it to her friend Bob, he explained her error, but when she simplified the problem correctly, she was amazed that she got the same result!
>
> Find some other numbers that have the same property.

This problem is from the Math9-12 project, a research-based initiative tasked with developing interesting and sophisticated mathematical activities for high school students. The project is led by Peter Taylor of Queens University. You can find further information on the initiative and check out the current projects on their website, www.math9-12.ca.

I tried this problem with my Grade 9 class. Professor Taylor came by to observe the students as they interacted with the problem. My students attacked the problem by either using trial and error or by trying to see some pattern in the given example. A few noticed that if they calculated the product, they got $4 \times \frac{4}{3} = \frac{4\times 4}{3} = \frac{16}{3}$. When they expressed the other number as an improper fraction, they got $4\frac{4}{3} = \frac{3\times 4+4}{3} = \frac{16}{3}$.

Noticing that the whole number and the numerator were equal and that the denominator was one less than the numerator, it wasn't long before some of the following expressions were discovered:

$$3\frac{3}{2} = 3 \times \frac{3}{2}, \qquad 5\frac{5}{4} = 5 \times \frac{5}{4}, \qquad 6\frac{6}{5} = 6 \times \frac{6}{5}, \ldots$$

When asked to explain why the pattern worked, however, the students struggled. One student was eventually able to describe well in words why the pattern worked. This student noticed that in the product, the numerator of the result will be a number multiplied by itself. In the mixed fraction, to get the numerator for the equivalent improper fraction you multiply a number by the number one less than itself, then add the same number, which yields the same result. For example, in the case of $7\frac{7}{6} = 7 \times \frac{7}{6}$, the numerator of the left side would be $6 \times 7 + 7$, but if you have 6 groups of 7 and you add another 7, you get 7 groups of 7 or 7×7, which is equivalent to the right side of the equation.

Even though I knew that the student understood why the problem worked, the other students in her group had a hard time following her reasoning. I asked them, and the other groups, if they could represent their solution algebraically. Many of the groups wrote their pattern as $x\frac{x}{x-1} = x \cdot \frac{x}{x-1}$ (meaning $x + \frac{x}{x-1} = x \cdot \frac{x}{x-1}$). At this point, they were able to use algebra to show that the expressions on either side of the equation simplify to the same expression and are thus equal. Having done so, they seemed more satisfied that they understood why it worked.

The intention of the exercise was to give students an opportunity to work with algebraic modelling. As such, let's rewrite the problem as

$$a + \frac{n}{d} = a \times \frac{n}{d},$$

where a, b, and n are all positive integers. We can find a common denominator, and the equation becomes

$$\frac{ad + n}{d} = \frac{an}{d},$$

hence

$$ad + n = an,$$

or $ad + n - an = 0$. The unfortunate thing about this equation is that we have three unknowns and only one equation. We could pick some numbers at random for two of the three variables and solve for the third, hoping to get an integer. If we are going to do that, might as well use some software to help us out.

I created a spreadsheet using Google Sheets to attack the problem. You can access it here: https://docs.google.com/spreadsheets/d/1ooYWqgHBKlyyA79-C0Jn974d1lLw0dWKUvj23DdLnGM/edit?usp=sharing. (If you would like to edit the spreadsheet, click on "File," then "Make a Copy" in the menu bar.)

The sheet has values for the numerator and the denominator in the rows and columns, respectively. The cell B1 contains the value of a and can be changed to see different solutions. In each cell below the denominators and to the right of the numerators, the value of $ad + n - an$ is calculated. If the result is 0, the cell is coloured red. Figure 1 shows part of the sheet using 3 for the value of a.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P
1	a =	3			d (denominator)											
2	(whole number)			1	2	3	4	5	6	7	8	9	10	11	12	13
3			1	1	4	7	10	13	16	19	22	25	28	31	34	37
4			2	-1	2	5	8	11	14	17	20	23	26	29	32	35
5			3	-3	0	3	6	9	12	15	18	21	24	27	30	33
6		n	4	-5	-2	1	4	7	10	13	16	19	22	25	28	31
7		numerator	5	-7	-4	-1	2	5	8	11	14	17	20	23	26	29
8			6	-9	-6	-3	0	3	6	9	12	15	18	21	24	27
9			7	-11	-8	-5	-2	1	4	7	10	13	16	19	22	25
10			8	-13	-10	-7	-4	-1	2	5	8	11	14	17	20	23
11			9	-15	-12	-9	-6	-3	0	3	6	9	12	15	18	21
12			10	-17	-14	-11	-8	-5	-2	1	4	7	10	13	16	19
13			11	-19	-16	-13	-10	-7	-4	-1	2	5	8	11	14	17
14			12	-21	-18	-15	-12	-9	-6	-3	0	3	6	9	12	15
15			13	-23	-20	-17	-14	-11	-8	-5	-2	1	4	7	10	13

Figure 1: Spreadsheet with a = 3

The spreadsheet reveals some interesting patterns. First, looking at the entries in any row or column, you will find arithmetic sequences (can you see why?). Also, the solutions (highlighted in red) seem to follow a pattern within the table: in particular, each solution is located 3 spaces below and 2 spaces to the right of the previous one. Extracting the solutions allows us to write the following equations in the original form:

$$3 + \frac{3}{2} = 3 \times \frac{3}{2}, \quad 3 + \frac{6}{4} = 3 \times \frac{6}{4}, \quad 3 + \frac{9}{6} = 3 \times \frac{9}{6}, \quad \cdots$$

which, upon a second glance, we realize are all the same solution. We can also see why this is true using algebra: If we go back to our original equation and replace $\frac{n}{d}$ with $\frac{kn}{kd}$ for some integer k, after the same type of rearrangement, we end up with $kad + kn = kan$, which is equivalent to $ad + n = an$. This tells us that we only need to consider reduced fractions.

Since this was supposed to be an exercise in algebraic modelling, let's see what happens when we "massage" the equation $ad + n = an$. Rearranging and solving for n yields

$$n = \frac{da}{a - 1}$$

At first sight, this doesn't seem to be very helpful. We could use a spreadsheet to come up with some solutions, but let's see what we can figure out algebraically. Since all of these numbers are positive integers, the above equation tells us that $a-1$ must be a factor of da.

Now, using a common mathematical bit of sleight of hand, we can add 0 to the numerator, but render it in a more useful form. We get

$$n = \frac{da}{a-1} = \frac{d(a-1+1)}{a-1} = \frac{d(a-1)}{a-1} + \frac{d}{a-1} = d + \frac{d}{a-1}$$

Since all of the values are integers, $a-1$ must be a factor of d, which means that $\frac{d}{a-1}$ must also be a factor of d, which in turn means that it will also be a factor of n (why?). However, we also know that our fraction was reduced, which means that the *only* factor n and d share is 1. Thus

$$\frac{d}{a-1} = 1,$$

which leads to $d = a-1$ and $n = d+1 = a-1+1 = a$, which is really just the pattern $x + \frac{x}{x-1} = x \cdot \frac{x}{x-1}$ that my students discovered.

If we had instead solved for d, we would get

$$d = \frac{n(a-1)}{a},$$

which leads to the same solutions in is similar way.

Finally, if we had solved for a, we would get

$$a = \frac{n}{n-d}.$$

Thus $n-d$ must be a factor of n, which in turn means that it must also be a factor of d. We can see this if we let $k = n-d$. Since k is a factor of n, we can write $n = kN$ for some integer N. Substituting this back into the equation for k yields

$$\begin{aligned} k &= n-d \\ k &= kN-d \\ d &= k(N-1) \end{aligned}$$

so k, which is equal to $n-d$, is a factor of d. Since we are only considering reduced fractions, the only factors that the numerator and denominator will have in common is 1. This will lead to the same solutions we have already discovered.

In the teachers' manual that accompanies the problem, the problem is modelled as

$$a + b = a \times b,$$

where a is a positive integer and b is a fraction. This leads to the same solution, but in a more efficient way. This shows us that not all mathematical models are equal. To see this for yourself, you may wish to work through the above model as well as through some of the other families of problems related to the task. For example, find more examples of equations similar to

$$\sqrt{3\frac{3}{8}} = 3\sqrt{\frac{3}{8}}.$$

Have fun!

And now for some homework:

> A tessellation is a tiling of the plane using one or more geometric shapes with no overlaps or gaps. Dutch artist M.C. Escher has many well-known pieces that consist of tessellations. Since tessellations can involve all kinds of complex shapes, we will concentrate on easy ones: regular polygons. Find as many configurations as you can of regular polygons that tile the plane without leaving gaps and without overlapping. You may use one or more different shapes in your tessellation.

Until next time, happy problem solving!

Shawn Godin teaches and is a department head at Cairine Wilson Secondary School in Orleans, Ontario. He strongly believes in the central role of problem solving in the mathematics classroom. He continues to be involved in mathematical activities: leading workshops, writing articles, working on local projects and helping create mathematics contests.

Spotlight *on the* **Profession**

In conversation with Dan Meyer

In this monthly column, we speak with a notable member of the mathematics education community about their work and their perspectives on the teaching and learning of mathematics. This month, we had the pleasure of speaking with Dan Meyer.

Dan Meyer taught high school math to students who didn't like high school math. He has advocated for better math instruction on CNN, Good Morning America, Everyday With Rachel Ray, and TED.com. He earned his doctorate from Stanford University in math education and is the Chief Academic Officer at Desmos where he explores the future of math, technology, and learning. He has worked with teachers internationally and in all fifty United States. He was named one of Tech & Learning's 30 Leaders of the Future. He lives in Oakland, CA.

First things first, thank you for taking some time out of your busy schedule for this conversation!

As Chief Academic Officer at Desmos, you spend a great deal of time considering the affordances of digital technology for the teaching and learning of mathematics, and designing tools that tap into these affordances.

The phrase 'online learning tools' evokes a diversity of conceptions and misconceptions; I wonder if you could address some of the latter. First, you have written that "the online medium is fundamentally connective and yet students often report feelings of social isolation" (Meyer, 2015a, p. iv). However, you have also argued that well-designed online tools can promote dialogue and collaboration, rather than isolation and individualization. How so?

For starters, whether we're in the high-tech or low-tech space, if students are doing interesting, creative work, the teacher has the opportunity to gather it purposefully and turn it into a conversation. So we have to start by giving students more interesting online work to do than multiple choice and numerical response. That kind of work is useful for some purposes, but as far as dialogue and conversation goes, it's bunch of wet twigs that won't start a fire.

Second, we have to realize that when students do something interesting digitally, they often try to share it. And that sharing stems from a natural interest in exhibiting something they're proud of, but it's also an interest in learning. In many cases, it's an interest in learning what their peers think about a video or photo they've taken. Will it get lots of likes or shares or retweets? Will it make someone laugh?

> "Whether we're in the high-tech or low-tech space, if students are doing interesting, creative work, the teacher has the opportunity to turn it into a conversation."

We don't exploit that interest in exhibition and learning with digital math tools. You create dull work and you share it with nobody but the machine. At Desmos, we seek to give students interesting work and put them in places to share it with each other.

Another fear related to online learning tools is that we are moving towards the obsolescence of teachers, who will be replaced by online programs that can provide instant feedback and the opportunity to move at your own pace (Khan Academy, for example, is viewed by some as an early glimpse into this future). What's your position? In your view, what role does the (human) teacher play in the digital age?

At Desmos, we design our activities with the assumption that a knowledgeable teacher is in the room. We can't realize our highest aspirations for student learning without teachers. We want to ask students questions that machines can't easily grade (written responses and sketches of relationships, for example) and we want students to see connections and structure in the class's thinking that machines don't know how to generate. We need teachers looking at all the work students are doing on our dashboard and thinking about the kinds of questions they'll ask about it. We envision productive human-computer partnerships of the sort we saw in the experiments of the late 1990s, where humans and computers working together in a chess match outperformed humans or computers on their own.

Related to your interest in supporting the teaching and learning of mathematics through digital technology is your interest in improving the teaching of mathematical modeling. You have written that you studied mathematics as a child and mathematics education as an adult because of powerful experiences you had of using mathematics as a model for the world around you, and would like students to have similar experiences (Meyer, 2015b). However, as you write, "modeling with mathematics […] [is] one of the practice standards most in need of explication. Five different teachers may have five different understandings of its meaning." (2015b, p. 579).

What do you mean by mathematical modeling? How do your Three-Act Tasks, a framework which you introduced on your blog in 2011, engage students in the process of mathematical modeling?

Modeling is a constellation of verbs that include identifying essential information to a question, formulating a model that structures that information, using the model, interpreting what the model tells you, and validating your interpretation back in the world. It's commonly seen as a cycle, where your validation underperforms your expectations so you circle back to your earlier assumptions and start again.

Students get very few of those experiences, even from problems in textbooks that are labeled "modeling."

The 3-Act Math project is not the final answer on mathematical modeling but it offers students some uncommon experiences. It uses digital media—photos and videos—to bring the world in from outside the classroom in ways that are more evocative than reading a text description on paper. A prismatic water tank filling up slowly—to take one example. It doesn't explicitly state any given information, giving students the chance to think about what's essential. And it also shows the answer to a question, allowing students to validate their modeling work more meaningfully than through an answer in the back of the book.

In Meyer (2015b), you argue that textbooks typically fall short of engaging students in mathematical modeling, despite their claims of doing so. Fortunately, you are not only interested in critiquing current resources—you have been working hard to improve them. You wrote in 2015: "I need a question to carry me through my thirties and I can't think of a better one than, 'What does the math textbook of the future look like?'" (Meyer, 2015c).

Beyond the missed opportunities to engage students in all of the processes of mathematical modeling, what are other critical shortcomings of many of today's math textbooks? What is your (working) vision for how the "textbook of the future" will address these shortcomings and better support students' learning of mathematics?

The textbook of the future will need to offer students provocative encounters with mathematics that are impossible on paper. For example, on paper, we can present a scenario and ask students to sketch the relationship between two variables in the scenario. Say, the height of an airplane over time—rendering the world into math, basically. What's impossible on paper and possible on computers is to then render math back into the world, to take the student's sketch and show the airplane height based on that sketch. We need loads and loads of encounters with math just like that—provocative of mathematical thought, and impossible on paper.

> "The textbook of the future will need to offer students provocative encounters with mathematics that are impossible on paper."

We also need to take advantage of the digital media's capacity to connect people together. A digital textbook should display your thoughts to me and my thoughts to you. Digital media has mutability that paper doesn't—so if you feel like my definition of a proportional relationship or yours has advantages over the textbook's, you should be able to add it into your textbook permanently. If you feel like you could take a more interesting photo representing a rhombus than the one in the textbook, you should be able to add it.

This is the low-hanging fruit. I have no idea yet what's higher up the tree until I clear what I can see first.

Lastly, I want to ask you about the affordances of digital tools for teacher *learning.*

You have often discussed the phenomena of blogging among mathematics teachers and the growing, self-driven community known as the Math Twitter Blog-o-Sphere (MTBoS), which connects thousands of mathematics teachers around the world who blog about their teaching practice and connect over Twitter under the hashtag #MTBoS. As Judy Larsen writes, "it is evident that they *spend hours writing publicly about their daily practice, posting resources, and sharing their dilemmas with no compensation and no mandate" (2016, par. 6). To outsiders, this may be surprising: "If you had to go back in time and bet that one group of teacher bloggers would break out in these amazing spasms of collaboration, admit that* math teachers *wouldn't have been your first or second guess" (Meyer, 2016).*

In your view, why are mathematics teachers particularly interested in blogging and connecting online, and what draws you, personally, to blogging and the MTBoS community?

I have no idea why math teachers have emerged as the most productive community of educators online—an opinion I don't think is controversial even among other kinds of educators. The sky is blue. Water is wet. Math teachers own Twitter. I've been a contributor for over ten years and I'm still confused. Happy about it, but confused.

> "The sky is blue. Water is wet. Math teachers own Twitter."

In those ten years, I've participated on Math Teacher Twitter for all kinds of different reasons. I needed resources. I needed community. I needed to process my thoughts in writing. I needed an audience in order to do my best writing. The consistent theme in my participation is the fact that my thoughts always seem perfect to me until they escape the vacuum seal of my brain. Once they're out in the world, in a blog post or a tweet, that's when I realize how much work they need. I can't get that feeling any other way.

Interviewed by Ilona Vashchyshyn

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Using the 5 Practices in Mathematics Teaching [1]

Keith Nabb, Erick B. Hofacker, Kathryn T. Ernie, & Susan Ahrendt

What might effective mathematics instruction look like if we were to see it? Engaging students with "challenging tasks that involve active meaning making and support meaningful learning" (National Council of Teachers of Mathematics [NCTM], 2014, p. 9) is one possible description. Staples in our classrooms include problem solving with cognitively demanding tasks, working in teams to formulate and solve problems, communicating mathematically through written and spoken channels, and critiquing or assessing the work of others. This article highlights three of the eight Mathematics Teaching Practices (MTP) published in NCTM's Principles to Actions: Ensuring Mathematical Success for All (2014, p. 10): facilitating meaningful mathematical discourse (MTP 4), posing purposeful questions (MTP 5), and eliciting and using evidence of student thinking (MTP 8).

In this article, we have several objectives. We open with a brief discussion of the meaning of the term active learning, and we discuss the five practices (Smith & Stein, 2011) as a particularly illuminating model. The five practices offer a powerful framework that we have used to activate our mathematics classrooms. Next, we share two vignettes of classroom learning from a first-year calculus class at a small university. Note that such teaching and learning experiences span all levels, K–16. The article's focus is on the implementation and management of active instructional practices, irrespective of mathematical content. Thus, we encourage the curious reader to join us in this experience (even if you do not teach calculus). The article closes with examples of student feedback from having experienced active learning in their college mathematics class.

Active Learning and the Five Practices

Although many definitions of "active learning" exist, most describe the same core qualities, regardless of the discipline or environment in which they are used. As early as 1991, active learning was described as "involving students in doing things and thinking about what they are doing" (Bonwell & Eison, 1991, p. 5). In a calculus class recently taught by the lead author, active learning strategies were implemented in which students were problem solving, discussing, and explaining their results to their classmates. A typical fifty-minute class ran as follows:

1. Preclass Phase

Students arrived to class "prepared" through a short preclass reading. The purpose of this reading was to cover

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fundamental principles, notation, and other ideas so that students could immediately engage with the content and with one another.

2. Problem Solving and Group Discussion

Once in class, students undertook a cognitively demanding task. Students worked in pairs or occasionally in groups of three, sharing ideas and talking about the problem. Eventually, they provided a solution on a mini whiteboard. Whiteboards were chosen as a display tool because they symbolize a common sharing space for the group's efforts and they proved suitable to assimilate group thinking.

Table 1. Descriptions of the Five Practices

Practice	Description	Question(s) a Teacher May Ask Himself or Herself	Miscellaneous
0. Identifying the Goal or Objective	Identify the specific goals of the lesson before class.	1. What do I want students to know and learn? 2. How should they know it?	Find and develop rich mathematical tasks where students may easily gain entry but from which interesting and relevant mathematics is likely to emerge.
1. Anticipating	Teacher predicts how students will solve the problem.	1. What will students do? 2. How will they do it? 3. What misconceptions are likely?	The teacher should solve the problem using a variety of strategies. Doing this allows the teacher to interpret a solution that was not anticipated more easily.
2. Monitoring	Teacher identifies the strategies used by visiting with groups, and answering and asking questions. Teacher begins documenting who is doing what.	1. What are students doing? 2. What strategies are being used?	If a group has misconstrued the problem, the teacher may wish to steer those students back on course.
3. Selecting	Teacher determines which groups should share their work.	1. Why should this group's work be showcased? 2. Why might other work not be shared?	This selection is driven by the goals and objective of the lesson (Practice 0).
4. Sequencing	Teacher determines a specific sequence that makes pedagogical sense. Those selected will present and discuss their work in this predetermined order.	1. What presentation order makes sense? a. Informal to formal? b. Simple to sophisticated? c. Common to unusual? 2. Should misconceptions be	The sequence should allow students to see connections from one group's solution to the next and offer opportunities for evaluating and critiquing work.

		addressed immediately or later?	
5. Connecting	Teacher directly makes connections in the approaches discussed or indirectly makes connections through questioning/focusing.	1. What is the story I want to tell with student work? 2. Are there other ideas that should be discussed—ideas that did not appear in students' efforts?	Student work is used to meet the goal of the lesson. A student or group may ask about a method not publicly shared and the teacher may have further opportunities to connect.

3. Whole-Class Discussion

Students shared their solutions with the class. Some group members explained their work at the front of the room; others explained it from their seats (a classmate from the group elevated the whiteboard for peers to see). Still others preferred the projection of the whiteboard contents on the document camera for easy whole-class viewing. These "sharing sessions" were the primary vehicle used to teach the day's content and meet the goals of the lesson.

A fifty-minute block had enough time to orchestrate two or three tasks (and their solutions), depending on the nature and content of the tasks. The details of how class was conducted are described in *5 Practices for Orchestrating Mathematics Discussions* (Smith & Stein, 2011), which succinctly captures a way to bring social interaction and active learning to mathematics classrooms. Although the book focuses specifically on K–12 levels, we feel that with suitable tasks, the practices can be a successful instrument at the college and university levels. The five practices are the following: (1) Anticipating, (2) Monitoring, (3) Selecting, (4) Sequencing, and (5) Connecting.

Smith and Stein contend that Planning/Goal Setting could be called "Practice 0," as this is something teachers need to do before orchestrating a productive discussion. Table 1 summarizes the five (or six) practices and describes salient characteristics of the implementation of each.

The table indicates various stages of teaching implementation. Practices 0–1 happen before the class meets, whereas practices 2–5 indicate active learning. In particular, practice 5 is the only part of the framework that—to an outsider—seems like "teaching." Students are the key players in the learning process (practices 2–4), and then once again when solutions are displayed and discussed (practice 5).

> "An essential piece of implementing the five practices is using high-level tasks. Groups are unlikely to "discuss" the mathematics if a task is too straightforward."

Before sharing classroom vignettes, some notes are in order. First, an essential piece of implementing the five practices is using high-level, cognitively demanding tasks. Groups are unlikely to "discuss" the mathematics if a task is too straightforward. The task should have a low entry point for engagement, a high bar for success, and be amenable to different approaches—all while meeting the specific goals of the lesson. This is a nontrivial cocktail of characteristics. We offer some resources at the end of this article to let readers know where to find mathematically rich tasks and how to repurpose "cookbook" tasks into more meaningful experiences (practice 0). The richer the task, the more scenarios the teacher will likely need to anticipate (practice 1).

Second, pedagogy of this sort embraces using student work to teach mathematical content. Practices 3–5 are manifestations of student work, so trusting what students are capable of producing is an absolute necessity. This phenomenon is not new to mathematics instruction (e.g., Rasmussen & Marrongelle, 2006), but it is far from the norm in tertiary education. Third, it is important to emphasize a classroom culture that values mistakes and learning from them. This pedagogy allows one to display common errors, build knowledge from these errors, and then connect this knowledge to valid mathematics. Finally, the five practices are not the same as a "show and tell" exhibit of student work (Smith & Stein, 2011). The Selection stage is carefully aligned to the goals of the lesson, and Sequencing is purposely done to make explicit the Connection phase for students. Thus, "more" is rarely synonymous with "better." The quantity and quality of the solutions should facilitate a productive discussion, which takes practice and skill on the teacher's part.

> "It is important to emphasize a classroom culture that values mistakes and learning from them."

Classroom Vignettes Using the Five Practices

In this section, we share classroom vignettes from two different units of first-semester calculus (one on limits and one on applications of the derivative). Our purpose is to highlight the similarities and differences of using the five practices with different types of problems and different types of student responses. As we emphasized earlier, teacher moves are situational and vary depending on the goals designated for the day's lesson. We zoom in on the practices of Selecting, Sequencing, and Connecting because these practices examine what work the teacher chose, why he or she chose it, and how this directed a productive mathematical discussion.

Vignette 1: Limits

The task given to groups was as follows:

TRUE or FALSE: If $f(x) < g(x)$ for all $x \neq a$, then $\lim_{x\to a} f(x) < \lim_{x\to a} g(x)$.

Justify!!

The goal of the task was for students to internalize that "operating" on a true statement with a limit may alter its truth value. A secondary goal supports the well-known fact that $f(a)$, if it exists, has no bearing on

$$\lim_{x\to a} f(x),$$

should the latter exist. This task is considered fairly complex, as it wraps these ideas into one simple true/false statement, and students are asked to defend their position. The whiteboards that were selected and sequenced (practices 3 and 4) are seen in Figure 1.

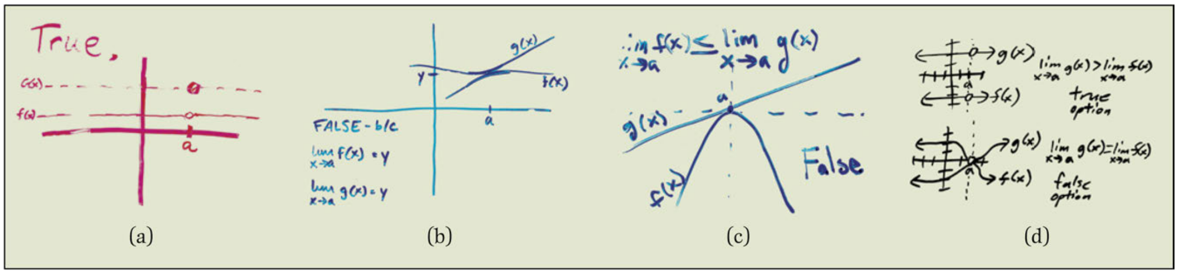

Figure 1: Group solutions were presented to the class in the order a, b, c, and d

Board 1 (see Figure 1a) was chosen to start the discussion for two reasons. First, two groups had precisely this response and thought that if $g(x)$ was "higher" than $f(x)$, then this relationship should remain true for the limit. Second, showcasing this board allowed the teacher to ask a pointed question, such as, "Are other pictures possible?" Students were quick to suggest making f and g closer to each other, which was a perfect segue into board 2 (see Figure 1b). This board was shared next because this group had grasped the basic principles but were not confident about their answer. An observer can see that the group engaged in much discussion of the values of f and g near $x = a$ (note the heavier lines there). Even though they had a valid response and two limit statements as added support, the group members were uncertain what was going on at $x = a$. They graciously confirmed this uncertainty with the class.

Board 3 (see Figure 1c) came next, as this group claimed "false" by considering the original statement in the problem,

$$\text{“}\lim_{x\to a} f(x) < \lim_{x\to a} g(x)\text{,”}$$

and modifying it to read

$$\text{“}\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x)\text{.”}$$

This not only justifies the falsity of the statement but also is the first of the three boards to use mathematically precise language aimed specifically at the original statement. Finally, board 4 (see Figure 1d) was shared; it encapsulates much of the work displayed on boards 1 and 2, refines board 3, and opens the door to an important mathematical discussion (practice 5: Connecting). One sees immediately the *two* answers this group provided— one analogous to the incorrect work on board 1 and one that challenges board 3 in that neither function need be defined at $x = a$. Group 4's incorrect work further supports the need to address the misconception in board 1, and their *correct* work illuminates the second goal of the lesson—that a function's value need not connect to its limit (should either exist).

The discussion that followed concerned the nature of mathematical truth: What does it mean for a statement to be true? Although students in group 4 thought both options were plausible, the members of group 1 explained to the class that it would take just one example to establish falsity. Group 1 members admitted they overlooked this situation before choosing to speak about it. Thus, although group 1 opened the Connecting stage with incorrect work, these students were, in fact, the ones driving the discussion of mathematical work on board 4. An unforeseen byproduct of this dialogue was a pointed discussion of two fundamental ideas that permeate all mathematical work—(1) needing proof to establish truth, and (2) generating a counterexample to establish falsity. Although this discussion was not one the teacher had anticipated, such welcome additions proved helpful in future discussions.

Vignette 2: Derivatives and Velocity
The task here read as follows:

> A machine is causing a particle to move along the x-axis so that its position at time t is given by $x(t) = (t - 4)^2$, where t is in seconds.
>
> (a) What is the particle's velocity at $t = 2$?
>
> (b) The machine stops suddenly at $t = 3$, releasing the particle. As the particle continues, where will it be 5 seconds after the machine stops? Explain your thinking.

The goal of this task was to allow students opportunities to apply the principles of position, velocity, and acceleration to solve problems involving change. Additionally, it was hoped that students would (1) be drawn explicitly to velocity as having both magnitude and direction and (2) negotiate and confirm specific assumptions (e.g., a frictionless environment) prior to solving the problem. Work was shared with the class (see Figure 2) after adequate time was given to produce a solution.

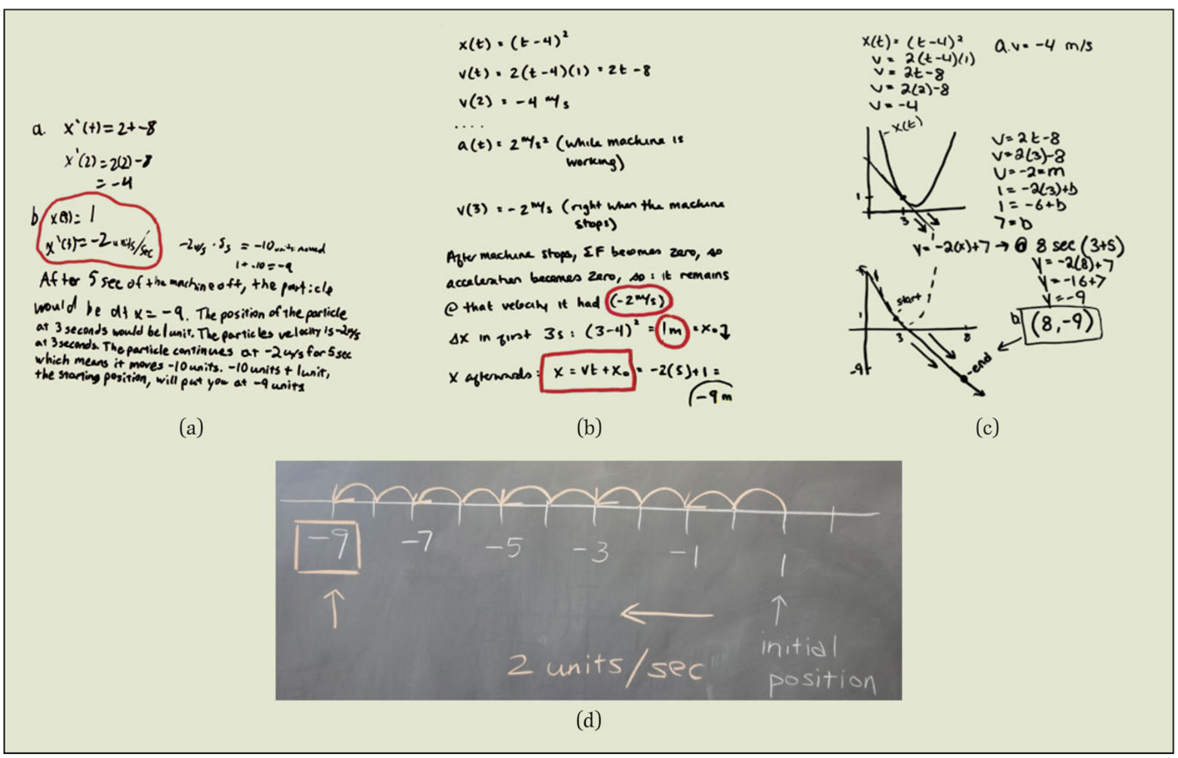

Figure 2: The teacher provided a solution (d) on the basis of a student's suggestion.

The rationale for the selection and sequence (practices 3 and 4) was as follows. Board 1 (see Figure 2a) started the discussion since this was both the most common solution and the highest priority with regard to anticipation. When a group member explained the group's work—specifically that "the particle continues at –2 units/sec for 5 sec," the student was clear in articulating what this meant: The particle was moving to the left, and no external forces were acting on the particle. The teacher had anticipated an explanation because no formal prerequisite knowledge of physics (friction) was assumed. Board 2 (Figure 2b) followed due to its sophisticated mathematical nature. Although its contents are equivalent to board 1, it is more formal in notation

(e.g., $\sum F$ and Δx),

and it makes explicit the multiplicative relationship once the initial position is established (i.e., $x = x_0 + vt$). For those who may not have understood where this formula came from, this knowledge was built from board 1, in which the same phenomenon was explained in simple, arithmetic terms. The attention to sequencing (practice 4) was deliberate— mainly to highlight a hierarchy in student thinking and conventions in mathematics notation.

Board 3—an unusual gem—was selected next (see Figure 2c). Admittedly, the teacher initially questioned the validity of its contents, and not until a minidiscussion with the group members was he convinced of its correctness. The group interpreted velocity as the slope of the tangent line and used its fixed slope as an indicator of the particle being in a vacuum, similar to the first explanation. The group then let time equal 8 seconds in the tangent line equation and obtained –9. When the teacher asked what these numbers meant, one group member claimed that the line was indicative of a constant velocity and that the point (8,–9) was to be interpreted as the coordinate (time, position). Her defense was that the original graph showed time versus position, so by using a straight line, they were assuming a constant velocity and determining where something ended up at a later time—precisely the objective of part (b) of the task (practice 0). Given this explanation, practice 5 was in full swing. We had an illuminating discussion connecting the equation from board 2, $x = x_0 + vt$, to the well-known $y = mx + b$ used in board 3. One class member added, "They're both just saying that *new equals change times time plus the old*."

> "Class discussions were a function of *students' ideas*—paving the way to meaningful understanding."

As we were set to wrap up the task, one member of the class asked if we could just solve the problem by counting. This prompted the teacher to illustrate some work on the chalkboard (see Figure 2d). Students could see that by starting at $x = 1$ and moving in spurts of 2 units to the left, one could end up with the answer. This was prompted by the structure evident in $x = x_0 + vt$, and it supported the written explanation on board 1.

Discussion

The vignettes above were chosen to highlight two typical classroom discussions using the five practices in the teaching of calculus. Class discussions were a function of *students' ideas*—paving the way to meaningful understanding. As a small representative sample, below are three responses from members of the class in an anonymous, end-of-course evaluation:

- Classroom facilitated learning in a hands-on manner. Allowed students to test their knowledge as well as inspired critical thinking.

- I like how the professor put the class into groups to try and solve problems together with peers instead of constant presentation-style instruction.
- I liked the style the class was taught with. The emphasis on group work and small-group discussion helped me understand the material better than a straight lecture.

On the basis of the comments above, we see that students acknowledged the time that they were given to think, make meaning, and contribute to mathematical discussions. Meanwhile, the teacher received a steady flow of information vis-à-vis "How are my students doing?" Because the assessment of students and groups was embedded in classroom teaching, this feedback then guided the teacher for the next lesson.

> "Because assessment was embedded in classroom teaching, this feedback guided the teacher for the next lesson."

Classroom teachers have asked us such questions as, "How does a student take notes in this environment?" and "What if a student or group misses the point entirely?" These are thoughtful questions, and our answers provide evidence of a paradigm shift in our teaching. For example, students are supplied with a written record of each classroom discussion through photographs of the whiteboards, often embellished with teacher comments. Knowing this ahead of time, students are less concerned with "taking notes" and more likely to make meaning of the mathematics being shared. The written record also addresses the second question of missing the objective of the lesson. Should a student or group fail to understand the material or even miss a day of class, the written document informs the student what he or she missed and provides pictures (literally!) of classmates' work and examples of student thinking. Generally, it is a win-win scenario for all. Moreover, much of the discussion above easily transfers to both high school and junior high school audiences.

Conclusion

The classroom examples shared here demonstrate what the five practices might look like in any mathematics classroom. Students are at the center of the learning, and the teacher navigates the terrain to ensure equitable, meaningful, and deep discussions about important mathematics. We have reshaped and repurposed many of the courses we teach to reflect an atmosphere in which students ask, explain, and connect. Without a doubt, our students are the greatest beneficiaries of this change.

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Co-planning Mathematics Lessons: Preparing to Make Connections Between Students' Solution Strategies

Tina Rapke, Marc Husband, and Amanda Allan

> *"These learning opportunities were not only purposeful tasks for my students, but a wonderful learning experience for me to grow as an educator and a pleasant and welcome reminder to not be afraid to delve outside of my comfort zone and explore new ideas in my own learning and growth."*
>
> *– excerpt from a teacher reflection*

The value that the mathematics education community places on student-developed solution strategies is clear (e.g., National Council of Teachers of Mathematics [NCTM], 2014; Yackel and Cobb, 1998). The research suggests that effective teaching—that is, teaching that supports learning with understanding—involves eliciting and making connections between students' mathematical ideas (e.g., Carpenter & Lehrer, 1999; Leatham, Peterson, Stockero, & Van Zoest, 2015; NCTM, 2000; Smith & Stein, 2011). But how do we prepare to teach such lessons? Teaching that focuses on and is responsive to students' mathematical ideas has proven to be complex and difficult (see, e.g., Ball, Lubienski, & Mewborn, 2001; Kazemi, Franke, & Lampert, 2009). For one, as Jacobs and Empson (2016) explain, such teaching requires teachers to "actively engage with children's thinking without imposing their own thinking" (p. 2), which may require teachers to set aside their preferred solution strategy in order to listen to their students' ideas. Such listening should not necessarily focus on finding a solution strategy the teacher already has in mind, but rather on supporting connections between the strategies students share (Mgombelo, Peres Toledo, & Rapke, 2016).

> "Teaching that focuses on and is responsive to students' mathematical ideas has proven to be complex and difficult. How do we prepare to teach such lessons?"

Both teachers and learners, however, have experienced the difficulty of trying to make sense of ideas that are different from their own, not to mention trying to generate different solution strategies and connect them to ideas and strategies put forth by others. In this article, we argue that co-planning addresses this issue, and that the mathematical connections teachers make while planning together can be transferred to the classroom and used to support students in connecting each other's mathematical ideas. Teachers implementing co-planned lessons in their own classrooms will benefit from having a repertoire of diverse strategies, which can help to prepare them to become more responsive to student thinking. In this way, co-planning offers practical ways to incorporate current research suggestions into classrooms.

In what follows, we describe the co-planning practice of a group of elementary school teachers, detail the solution strategies that emerged during one particular co-planning session, and examine the teachers' reflections about their co-planning experiences. We then discuss the benefits of co-planning, which include deeper mathematical understanding, the potential to shift the ways in which we think about the role of a mathematics teacher, and bringing the joy back into lesson planning.

Description and details of co-planning

In line with John's (2006) proposition that lesson planning is a practice, we do not see lesson planning as simply, or primarily focused on, the creation of a document that describes a lesson. We see planning as a practice that can be shared, and that is enhanced by sharing. In the co-planning session that we detail below, three teachers from the same school (Rebecca, Tanis, and Tony—pseudonyms) came together with a curriculum leader (second author) to co-plan one lesson for Tony's Grade 5 class. The co-planning session was part of a larger ministry-funded project (see Lundy, 2016). At the time of the co-planning session, Rebecca taught Grade 7 and Tanis taught Grade 3/4. The co-planning session described here was motivated by the following Grade 5 curriculum expectation: "divide three-digit whole numbers by one-digit whole numbers, using concrete materials, estimation, student-generated algorithms, and standard algorithms" (Ontario Ministry of Education, 2005, p. 79). Once the teachers agreed on the expectation and a corresponding mathematics problem (248 divided by 8), they worked on anticipating 'student-generated algorithms' (an emphasis of their regional mathematics curriculum). The teachers began by sharing their own preferred strategies to solve 248 divided by 8, after which they anticipated other strategies their students might share. The teacher leader annotated the teachers' strategies on a whiteboard as they emerged. The annotated solutions allowed the teacher group to view the strategies as a whole, and supported the process of connecting the strategies based on similarities and differences.

> "We do not see lesson planning as simply creating a document that describes a lesson. We see planning as a practice that can be shared, and that is enhanced by sharing."

More specifically, our co-planning session consisted of the following stages:

- choose a curriculum expectation;
- find/craft an appropriate mathematics problem to target the expectation;
- anticipate students' preferred strategies to solve the problem;
- consider connections between the various strategies.

We decided to co-plan in this way because we believe it is beneficial for teachers to first generate and observe multiple solution strategies and then to work to connect these strategies in preparation for connecting the ideas students share in the classroom.

Anticipated student solution strategies

The co-teachers anticipated five solution strategies to the problem of dividing 248 by 8: halving, equal partitioning, skip counting and doubling, multiplying by a 'friendly' number, and the traditional division algorithm.

The halving strategy

One of the teachers remarked that many of his students often use a halving strategy when solving a division problem. Using this strategy, students might split 248 in half and continue halving until they noticed 8 groups of 31 (see Figure 1).

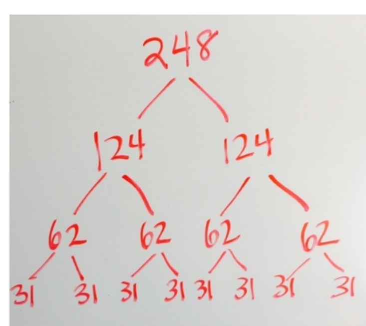


Figure 1

Equal partitioning

After the halving strategy was discussed, another teacher imagined students decomposing 248 into hundreds, tens, and ones (i.e., 248 = 200 + 40 + 8) and then coming to a solution by equally partitioning these numbers into eight groups. One teacher suggested that some students might prefer to work with the 100s because of their previous experiences with money and/or their knowledge that 100 is easily decomposed into 4 equal groups of 25. Combining these ideas, teachers thought that some students might use these understandings to observe that 200 can be split into two 100s, and therefore into eight equal groups of 25. Following this line of thinking, they anticipated that students would use the fact that 40 can be split into two 20s, and 8 into two groups of 4. They suggested that students might then notice that 20 can be split into 4 equal groups of 5s and 4 into 4 groups of 1s, ultimately concluding that 40 can be partitioned into 8 groups of 5s, and 8 into 8 groups of 1s. Teachers imagined that students might think about these partitions using the metaphor of distributing items equally into 8 boxes. Concluding this line of reasoning, a student might then claim that each of the eight boxes would contain one group of 25, one group of 5, and one 1—in total, 31 (see Figure 2).

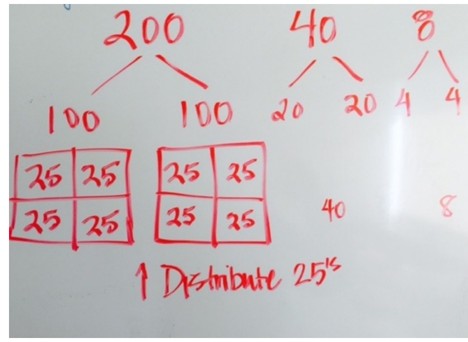
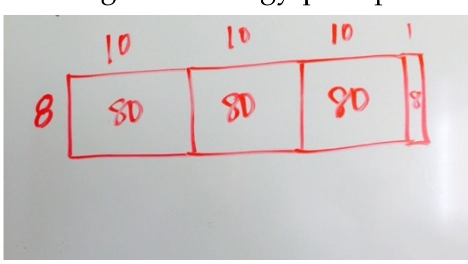


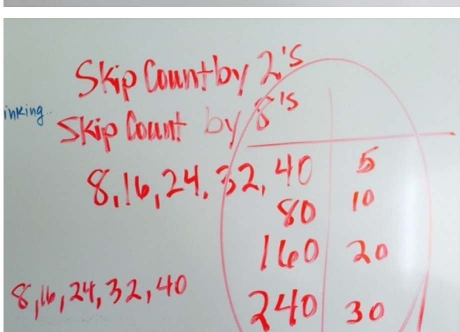
Figure 2

Anticipating solutions that begin with the whole (248) prompted a discussion about the relationship between division and multiplication. Our attention was drawn to how some students might prefer to solve the problem using multiplication. We anticipated two such strategies:

Skip counting and doubling

One teacher anticipated that some students would recognize 248 as an even number and therefore know that they could skip count by 2s all the way to 248. The mention of skip counting as a strategy prompted another teacher to comment on the exhausting work of skip counting by 2s, wondering if students who chose to skip count by 8s would eventually begin skip counting by still larger numbers. For example, skip counting by 8s five times gives 40 ($8 \times 5 = 40$). Students might pause and notice 40 as a friendlier number to count from. Instead of continuing to count by 8s, students might use this opportunity to double 40 to 80 and then essentially start to skip count by 80s instead of 2s. The alternative might be that skip counting would lead to (or involve) a continued doubling strategy. For instance, students would skip count to 40 and then double ($2 \times 40 = 80$) and double again to 160 ($2 \times 2 \times 40 = 160$). At this point they might abandon a doubling strategy and notice that one more 80 would bring them to 240. (See Figures 3 and 4.) Tony reacted to the emergence of this thinking: "Aha, see for me, this is proportional reasoning."





Figures 3 and 4

Multiply by a 'friendly' number

Building on the friendly number idea proposed during the discussion of skip counting and doubling strategies, the teachers anticipated students' preferences for multiplication involving friendly numbers such as 10. The teachers anticipated that some students might use 10 as a benchmark number and multiply until they got as close to 248 as possible. One teacher suggested that students who use this strategy will likely first estimate 8 × 10 and notice that 80 is too low. Students might continue by computing 8 × 20 = 8 × 10 + 8 × 10 = 160 and then 8 × 30 = 8 × 10 + 8 × 10 + 8 × 10 = 240. One of the co-planning teachers mentioned that at this point, students would likely notice that one more group of 8 (1 × 8) is required. We recorded this solution using an open array (a rectangular arrangement of rows and columns); see Figure 5.

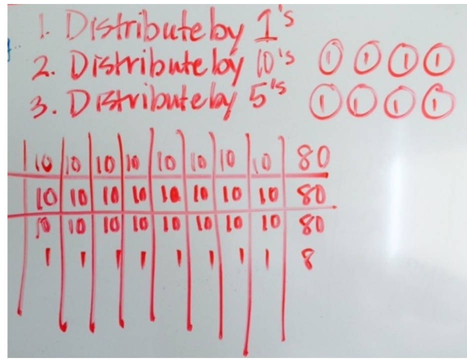


Figure 5

The traditional algorithm

Students' use of a 'standard' algorithm for division eventually worked its way into the conversation. (See Figure 6 for an example of the traditional algorithm.) One of the teachers shared that she did not remember how to use the algorithm. This prompted another teacher to share Arthur Hyde's (2009) description of 'Guzinta': As Hyde explains, many students only hear, in this case, "8 Guzinta 248," not understanding the strategy of using division to determine how many 8s are in 248. According to Hyde, while 'Guzinta' helps students cope with remembering the procedure, if taught in isolation, it does not promote understanding of the concept.

The 'Guzinta' anecdote not only provided the teachers with a laugh, it provoked them to consider how they could deepen their understanding of the division algorithm by connecting it to the different division strategies they had anticipated. In particular, the teachers used notions within the 'multiply by a friendly number' strategy to make sense of the algorithm, using the fact that 3 □ 8 = 24 to make sense of the 3 above the 4 (recorded in the tens place), which represents how many 80s are in 240.

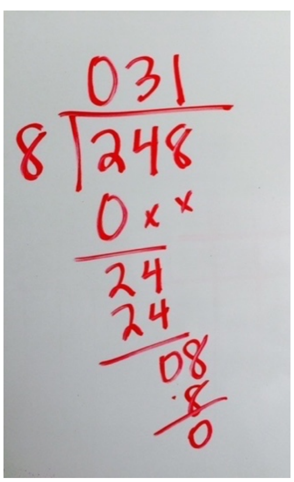


Figure 6

Further connections were made during discussions about the similarities and differences between the anticipated strategies. For instance, both the halving and equal partitioning strategies begin with the whole, which is then partitioned into smaller equal groups (these were referred to as whole-to-part strategies), whereas the strategies of skip counting, doubling, and multiplying by a 'friendly' number begin with the part and continue to make progressively larger equal groups (these were referred to as part-to-whole strategies).

Stepping back further, the teachers noticed that both whole-to-part and part-to-whole strategies involved forming equal groups—in the first case, splitting up 248 to get to equal groups, and in the second case, increasing the number of elements in the equal groups to get to 248. This connection of the strategies to a bigger mathematical idea (creating equal groups) ignited a discussion about personal preferences for either increasing from the part (8) to the whole (248) or decreasing from the whole (248) to the part (8).

Benefits of co-planning

We found that the co-planning sessions enhanced teachers' learning by a) re-conceptualizing teaching as responsive to student thinking, thereby preparing them to draw out connections between different students' mathematical thinking, and b) deepening their understanding of mathematical concepts through the process of making connections between strategies. Furthermore, we (the authors) and the teachers greatly enjoyed the time spent co-planning together. Our experiences suggest that co-planning can re-conceptualize not only teaching, but the planning process as well.

As evidence of the potential for a re-conceptualizing of teaching, we share the following excerpt from Tony's reflection on the session:

> "I have become more in tune not necessarily to what I want the students to learn, but rather, what am I expecting or predicting the students will respond to the learning. With this … preparation I am able to respond appropriately and provide an opportunity to steer or challenge the students' thinking further."

This reflection suggests that this teacher now believes that part of the teacher's role is to emphasize and respond to students' thinking. Tony indicates that this is a new way, to him, to conceptualize his role, as suggested by the phrasing "I have become."

Co-planning also offered teachers opportunities to deepen their understanding of mathematics. Specifically, in this case, it led Rebecca to "look beyond the obvious [standard way to solve a problem] and discover so many mathematical connections" between the strategies. Rebecca also remarked that while she had forgotten the traditional algorithm—"I don't even remember how to do it this way"—, co-planning allowed her to deepen her understanding by making sense of the standard algorithm through making connections to the other anticipated strategies.

> "Co-planning can reconceptualize lesson planning as a process, rather than a product, and bring enjoyment back into the planning of a lesson."

Finally, co-planning approached in this way—that is, as a practice focused on anticipating and connecting student strategies—reconceptualizes lesson planning as a process, rather than a product, and brings enjoyment back into the planning of a lesson for the teachers involved. Imagining what students' strategies might look and sound like seemed to ignite enthusiasm for the planning process, and many laughs were shared during the process (e.g., during the 'Guzinta' discussion). As further evidence of the fun teachers can have while co-planning, at one point during the session Tanis exclaimed: "Can we do this tomorrow!?"

Conclusion

Co-planning is a professional learning experience that offers many benefits, some of which have been outlined in this article. Indeed, we (the authors) attribute much of the growth in our own teaching practices to co-planning. However, we recognize that co-planning does take time and dedication, and that the day-to-day demands of the teaching profession can limit the opportunities for teachers to co-plan regularly and at the level of depth outlined in this article. While we were fortunate enough to receive funding from a larger ministry-funded project to pursue co-planning, we too have had to dedicate some of our own time to learning with and from one another. We hope that support for the practice of co-planning will continue to increase, especially as research reports (e.g., Lundy, 2016) strongly recommend more time to be provided for teacher collaboration (including co-planning).

We hope that we have convinced you, the reader, to experiment with co-planning in your own school. Try co-planning a lesson with a colleague teaching the same grade so that you too can experience the benefits, and then share your experiences with your colleagues. In doing so, we may inspire administrators and policymakers to create scheduled opportunities for teachers to work together on anticipating and connecting mathematical ideas so that they can help their students do the same.

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Intersections

In this column, you'll find information about upcoming math education-related workshops, conferences, and other events. Some events fill up fast, so don't delay signing up!

For more information about a particular event or to register, follow the link provided below the description. If you know about an event that should be on our list, please contact us at thevariable@smts.ca.

Within Saskatchewan

Workshops

Technology in Mathematics Foundations and Pre-Calculus
May 7, 2018
Saskatoon, SK
Presented by the Saskatchewan Professional Development Unit

This workshop is designed to have math foundations and pre-calculus teachers experience a variety of technology tools that allow students to represent and visualize mathematics concepts. Tools highlighted are useful for students to explore, learn, communicate, collaborate and practice in order to enhance their understanding of mathematics in secondary mathematics.

Head to www.stf.sk.ca/professional-resources/events-calendar/technology-mathematics-foundations-and-pre-calculus-0

Number Talks and Beyond: Building Math Communities Through Classroom Conversation
May 11, 2018
Saskatoon, SK
Presented by the Saskatchewan Professional Development Unit

Classroom discussion is a powerful tool for supporting student communication sense making and mathematical understanding. Curating productive math talk communities required teachers to plan for and recognize opportunities in the live action of teaching.

Come experience a variety of classroom numeracy routines including number talks, counting circles, quick images and more. Take math conversations to the next level by strengthening your skills as a facilitator of classroom discourse and student thinking.

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Making Math Class Work
May 14, 2018
Saskatoon, SK
Presented by the Saskatchewan Professional Development Unit

Math classrooms across Saskatchewan are increasingly complex and diverse. Meeting everyone's needs can be daunting, even with all of the instructional strategies and structures available to teachers. Number Talks, Guided Math, Rich Tasks, Problem Based Learning, Open Questions, High Yield Routines are just some of the strategies available to teachers, but where to start? Come work collaboratively to problem solve how to make math class work for you and your students.

Head to www.stf.sk.ca/professional-resources/events-calendar/making-math-class-work

Summer Initial Accreditation Seminar
July 9-13, 2018 (Saskatoon) / August 13-17, 2018 (Regina)
Presented by the Saskatchewan Professional Development Unit

Accreditation seminars are offered to enable qualified teachers to become accredited. Accreditation is the process by which qualified teachers are granted the responsibility of determining the final mark or standing of the students in a specified Grade 12 (level 30) subject or subjects. The Accreditation seminar provides an opportunity for teachers to challenge, extend, enhance and renew their professional experience with an emphasis on assessment and evaluation.

Saskatoon: www.stf.sk.ca/professional-resources/events-calendar/saskatoon-summer-initial-accreditation-seminar

Regina: www.stf.sk.ca/professional-resources/events-calendar/regina-summer-initial-accreditation-seminar

Summer Renewal / Second Accreditation
July 9-11, 2018 (Saskatoon) / August 13-15, 2018 (Regina)
Presented by the Saskatchewan Professional Development Unit

Accreditation seminars are offered to enable qualified teachers to become accredited. Accreditation is the process by which qualified teachers are granted the responsibility of determining the final mark or standing of the students in a specified Grade 12 (level 30) subject or subjects. The Accreditation seminar provides an opportunity for teachers to challenge, extend, enhance and renew their professional experience with an emphasis on assessment and evaluation.

Saskatoon: www.stf.sk.ca/professional-resources/events-calendar/saskatoon-summer-renewalsecond-accreditation

Regina: www.stf.sk.ca/professional-resources/events-calendar/regina-summer-renewalsecond-accreditation-seminar

Programs

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Start dates vary
Faculty of Education, University of Regina

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For more information, head to
www.uregina.ca/education/programs/certificates.html#ctesm

Online Workshops

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Presented by Education Week

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Past webinars: http://www.edweek.org/ew/webinars/math-webinars.html
Upcoming webinars:
http://www.edweek.org/ew/marketplace/webinars/webinars.html

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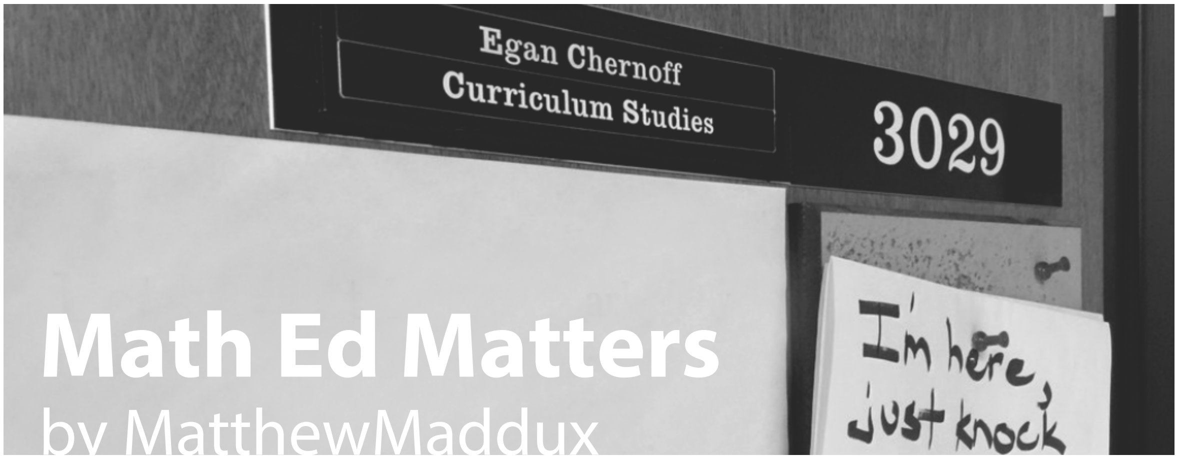

Math Ed Matters by MatthewMaddux *is a bimonthly column telling slightly bent, untold, true stories of mathematics teaching and learning.*

Precisely Innacurate: Putting the 'Ass' in Assessment

Egan J Chernoff
egan.chernoff@usask.ca

"I just don't see it happening, Egan." This was the then-devastating response from my Chemistry 12 teacher when I had asked him about my chances of pulling off an A (86%) by the end of the year. (This class grade would get boiled down to 60% of our total grade after we wrote the Provincial Exam, which accounted for 40% of our grade, but that's a conversation for another time.). I'll admit, my grade for Term 1 that year wasn't great. But it wasn't horrible. According my calculations, if I pulled off a near-perfect Term 2, that A that I wanted so badly wouldn't be impossible.

The grades for my Chemistry 12 class were based on labs, quizzes, and chapter tests. Getting good grades on the labs, for me, wasn't a problem, which left the quizzes and the chapter tests. Motivated by my teacher's comments and wanting to major in Chemistry at university next year, I studied ferociously for those quizzes and chapter tests. It paid off. I had nearly perfect results in Term 2 of Chemistry 12. My classmates were surprised, my teacher was definitely surprised and, if I'm being honest, I even surprised myself.

As was the case in Term 1, on the last day of class in Term 2 the teacher would bring each of us into that little room, the one that is found in the back of most Chemistry classes, to break down our grade on the computer. I was eager to hear my grade. Having kept meticulous track of my grades for every lab, every quiz, and every chapter test, I knew that I had done it. I had pulled off a near-perfect Term 2. Coupled with my grades from Term 1, it would be close, but I had secured my A for the year... or so I thought.

> *Dr. H.*: Great semester, Egan. Just great!
>
> *Egan*: Thank you, Sir.
>
> *Dr. H.*: As you can see on the screen here, you did great on the labs, as you did last term. But the big change was on your grades for the quizzes and the chapter tests. What happened?

Egan: Well, I was pretty motivated. I studied really hard this semester. I want to major in Chemistry and, one day, become a Chemistry teacher. Getting an A in your course would mean that I can take an advanced Chemistry course at university next year, because they consider an A in Chemistry 12 as equivalent to a first-semester university class.

Dr. H.: I see… Well, here's your grade for Term 1 [points to the computer screen]. Here's your grade for Term 2 [point again to the computer screen]. And, when you put the two grades together you get 85.499%, which means your final grade is 85% for my course. Well done!

Egan: Wait, what?!

Dr. H.: I'm sorry?

Egan: Well, I calculated my grade, too, and I got 86%.

Dr. H.: Well, let's look over the grades again to make sure we have the same numbers for everything.

[Each of the numbers for each the labs, quizzes and chapter tests are verified.]

Dr. H.: Well, I'm glad we have the same numbers. So we agree, then, on your grade of 85%.

Egan: Not exactly…

What happened next in the exchange between my Grade 12 Chemistry teacher and myself has stuck with me to this day. At the time, I was crestfallen. Devastated. Maybe you, too, have been either on the giving or the receiving end of the following exchange:

Egan: I'm sorry, Sir, but I don't understand... isn't that an A?!

Dr. H.: I don't see how. If we round 85.499 to the nearest one, you get 85.

Egan: Yeah, but if you round 85.499 to the nearest tenth or to the nearest hundredth, then you get 85.5. And if you round 85.5 to the nearest one, you get 86!

Dr. H.: Well, I can't go doing all that… Even if I did round to the nearest tenth or the nearest hundredth, like you said, 85.5 rounded to the tenth or the hundredth is still 85.5, not 86.

[After taking a moment to collect myself]

Egan: So you're telling me that I missed an A in your class by *one thousandth* of a percent?!

Dr. H.: What can I say, Egan. As you well know, numbers don't lie.

I vowed that day, right then, right there: Should I ever become a teacher, I would never do what had just happened to me. At the time, my vow was a volatile concoction of surprise, disbelief, and a semester's worth of hard work not coming to fruition, mixed with all the other feelings an 18-year old human being feels when things don't go their way. But contrary to most vows made by teenagers, as time marched on, support for my assertion started to appear everywhere I looked, both in my personal and my professional life.

As I soon came to realize, decisions can be made either by the letter of the law or by the spirit of the law. For example, a person who is speeding 30 km over the speed limit can, of course, be issued a ticket indicating that they were traveling 30 km over the limit. That's letter of the law. It is the purview of the officer, however, to take certain things into consideration when issuing said ticket. An individual who has never received a speeding ticket in their life may get a speeding ticket indicating that they were over the limit, but by a lesser amount than was recorded, thus easing an otherwise excellent driver's fine and demerit points. On the other hand, a perpetual speeder might get the book thrown at them. Either way, that's the spirit of the law. Examples are everywhere. And the more I saw it in the world around me, the more absurd I found it that the letter of the law had been applied to my Grade 12 Chemistry grade. But maybe I just wasn't looking at the situation through the lens of a professional.

> "If the spirit of the law was dominating in most spheres of daily life outside of school, why deny students of this reality while preparing them to live and work in the world beyond the school walls?"

During my first year as a math teacher, the first year I had to calculate grades for students, I was acutely aware of the percentages associated with different letter grades. Armed with my professional autonomy, I made sure that any borderline cases were dealt with in great detail. I would pore over the grades for the assignments and make absolutely sure that there were no calculation errors or missing grades standing in the way of a C+ being a B or a B being an A. After all, if the spirit of the law was dominating in most spheres of daily life outside of school, why deny students of this reality while preparing them to live and work in the world beyond the school walls? And yet, as I would realize later, I was still too aligned with the letter of the law. For example, if a student's grades averaged out to 72.00%, which equated to a C+, I couldn't figure out a way to justify changing that grade to 73%, which equated to a B. That is, until I learned an important lesson about the difference between accuracy and precision. It all started in the garden.

I am by no means a woodworker. With that said, I do have a few tools that cut wood, and a few others that put wood together. Working on a flower bed one year, I made sure to follow the Russian proverb that my father always followed, as did his father: Measure seven times; cut once. With a semi-expensive, 8-foot-long 4 x 8 in front of me, I made sure to measure once and lightly mark with a pencil the desired length on the board. Hearing my father's and grandfather's voices, I measured again. Sure enough, this time, the marking line was inside of my original line. With this visual discrepancy in front of me, I measured a third time, and the new marking line was closer to the original line than the second line drawn. For good measure, I repeated the process two more times. Surely, the measurement I was making was becoming more precise. And so, finally, armed with a composite line that was somewhere inside of the original line that I had drawn, I was ready to make my cut.

You've probably guessed what happened next. Armed with my precision cut, I made my way over to the insert the remaining piece of my flower bed—only to find that the piece I had cut was too short by no less (and no more) than four inches. Very precise—my careful measurements were within millimeters of one another—but very inaccurate. And when it comes to flower beds, precisely inaccurate doesn't cut it.

Days later, while I was grading papers, my very precise, but very inaccurate cut got me thinking back to the grade of 85.499% that I had received in Chemistry 12 and the grades that I had by then been giving out for years as a math teacher (to three decimal places, thanks to my digital gradebook). Despite my teenage vow, my notion of assessment accuracy was being confused with precision—that is, with a grade reported to a thousandth of a decimal point. I soon realized everything that would be required to accurately assess my students to a thousandth of a decimal point: Perfect assessment of perfect assignments and perfect quizzes and perfect tests. Assessment where I properly inferred every ambiguity in every answer to every perfectly-designed test question. Finally, many years after that last day of Chemistry 12, the insanity of it all sank in. I am not able to assess students to a thousandth of a decimal point. I am not able to assess to a hundredth of a decimal point. I am not even able to assess to a tenth of a decimal point. It's perhaps even a bit insane to think that I am able to assess to the nearest integer, although I had labeled students with them for years. I arrived at an uncomfortable realization: I was putting the ass in assessment! It's a hard pill to swallow in a world where my phone always knows my precise location in this world, but as I think more and more about *measurement accuracy*, I find that my *tolerance* is ever growing.

> "I had arrived at an uncomfortable realization: I was putting the ass in assessment!"

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