The Variable

Examining Mistakes *to* Shift Student Thinking

Spotlight on the Profession:
In conversation with Dr. Ilana Horn

What Makes a Function, Function?

Little Signs of Innumeracy

Saskatchewan Mathematics Teachers' Society

AFFILIATE
NATIONAL COUNCIL OF
TEACHERS OF MATHEMATICS
NCTM

In this issue

Cover Image

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Notice to Readers

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Notice to Contributors

The Variable welcomes a variety of submissions for consideration from all members of the mathematics education community, including classroom teachers, consultants, teacher educators, researchers, and students of all ages. Submitted material is assessed for interest, presentation, and suitability for the target audience.

Please submit articles by email to thevariable@smts.ca in Microsoft Word format. Authors should aim to limit submissions to 3000 words or less and include a photo and a short biographical statement of 75 words or less with their submission. Accepted contributions are subject to revision. Editors reserve the right to edit manuscripts for clarity, brevity, grammar, and style.

Call for Contributions

***The Variable* is looking for contributions from all members of the mathematics education community**, including classroom teachers, consultants, and teacher educators. Consider sharing a favorite lesson, an essay, a book review, or any other work that may be of interest to mathematics teachers in Saskatchewan. If accepted for publication, your article will be shared with a wide audience of mathematics educators in Saskatchewan and beyond.

We also welcome student contributions in the form of artwork, stories, interesting problem solutions, or articles. This is a great opportunity for students to share their work with an audience beyond that of their classroom, and for teachers to recognize students' efforts during their journey of learning mathematics.

All work is published under a Creative Commons license. **To submit or propose an article, please contact us at thevariable@smts.ca**. We look forward to hearing from you!

Ilona & Nat,
Editors

Saskatchewan Mathematics Teachers' Society

The Saskatchewan Mathematics Teachers' Society presents ...

#SUM2018
November 2-3, 2018

Who: K-12 teachers, coaches, consultants, coordinators, superintendents and directors

Where: Circle Drive Alliance Church

When: November 2-3, 2018

Cost: $165 (early registration) | $210 (regular)

Keynote Presenters

Lisa Lunney Borden
St. Francis Xavier University

Mary Bourassa
Ottawa-Carleton District School Board

AFFILIATE
NATIONAL COUNCIL OF
TEACHERS OF MATHEMATICS

Message from the President

W elcome back! I hope that you've had a rejuvenating summer, and that your first week back with students fills you with excitement for the year to come!

As much as I love summer (and I really do love summer!), I also love getting back into routines and the promise of a new year. This year, I have taken the extra accountability step of writing out my goals so I can look back on them periodically to adjust and keep myself on track. I'd invite you to do something similar so that we can check in collectively later on in the year and share stories of how we're making routines work for ourselves and our students in a whole variety of ways.

The Saskatchewan Mathematics Teachers Society certainly hopes that the Saskatchewan Understands Math (SUM) conference is beginning to become part of your fall routine. We are beyond excited to have Lisa Lunney Borden returning so that we can learn more about how to honor, support, and celebrate First Nations' ways of knowing in the mathematics classroom. Lisa will also be discussing the role that mathematics education plays in reconciliation and the work that we can do as mathematics educators.

This year, we are also excited to have Mary Bourassa join us as keynote speaker at SUM Conference. Mary is a classroom teacher in Ottawa and speaks passionately and concretely about the changes we can make as teachers to improve learning for students in our classrooms. She speaks from a place of honesty and experience about the challenges and successes in growing her own practice, as well as the benefits that her perseverance has had for students.

The only thing missing from our SUM lineup is you! Now is the time to get your session proposals in - we can't wait to add your name to the list of educators so we can learn from you! To remind the more reluctant among you, your everyday, 'normal' routine might just be someone else's "Aha!" moment. Take a moment to reflect on what you might share with your colleagues from around the province, and then make it happen! Head to our website (www.smts.ca/sum-conference/sum-call-for-proposals/) to submit your proposal before October 5.

Have a wonderful start-up, and we'll see you in November!

Michelle Naidu

Problems *to* Ponder

Welcome to this month's edition of Problems to Ponder! Have an interesting solution? Send it to thevariable@smts.ca *for publication in a future issue of* The Variable!

Primary Tasks (Kindergarten-Intermediate)

Lady Bug Spots[1]

Materials:
- ladybug printout (p. 28 of resource—see footnote)
- 5 counters or other manipulative

Directions: This game is played in pairs.

1. Place 5 counters on the ladybug.
2. Player 1 closes their eyes and Player 2 takes some of the counters off the ladybug.
3. Player 1 looks at the ladybug and determines how many counters Player 2 took off.
4. To check, players count the counters removed together.
5. Players take turns removing counters and determining how many were removed.
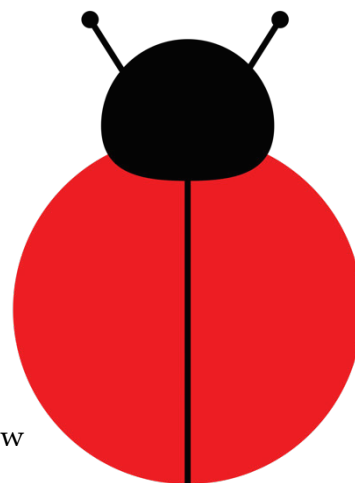
Extension: Use more counters as students develop proficiency with addition and subtraction.

Variation: Player 1 places counters on the ladybug. Player 2 determines how many need to be added to the ladybug to make five (says "add _____"), and places that number of counters on the ladybug.

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[1] North Carolina Department of Public Instruction. (n.d.). *Building conceptual understanding and fluency through games: Grade K.* Retrieved from http://maccss.ncdpi.wikispaces.net/file/view/Kgrade_GAMES.pdf/522022884/Kgrade_GAMES.pdf

Rectangles[2]
Materials:
- pair of standard dice
- gameboard consisting of a 20x20 grid (p. 77 of resource—see footnote)
- crayons or coloured pencils

Directions: This game is played in pairs.

1. In turn, each player rolls the dice. A player outlines and colors a rectangle on the gameboard to match the outcome. Example: a roll of 3 and 6 = a 3 x 6 rectangle or a 3 x 6 rectangle.
2. The player writes an equation to represent the total number of squares (area) in the center of the rectangle.
3. A player loses a turn when he or she rolls and cannot fit his or her rectangle on the gameboard. The game is over when neither player can draw a rectangle (*alternatively, decide on a specific number of turns to be played*). The winner is the player who has claimed the largest total area.

Variations/Extensions: Change the dimensions of the game board, the dice, or let each player have their own game board.

Number Tic-Tac-Toe[3]
Draw a tic-tac-toe board (see below). Instead of using X's and O's, use the numbers from 1 to 9. Each number may be used only once in a game. Take turns writing a number in a space. The goal of the game is to be the first to get the numbers in any row, column, or diagonal to add up to 15.
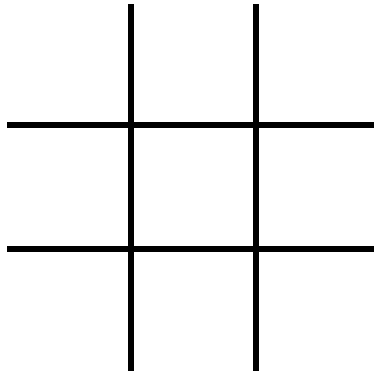
[2] North Carolina Department of Public Instruction. (n.d.). *Building conceptual understanding and fluency through games: Grade 4.* Retrieved from http://maccss.ncdpi.wikispaces.net/file/view/4thgrade_GAMES_8.22.14.pdf/593155854/4thgrade_GAMES_8.22.14.pdf

[3] *Adapted from* Burns, M. (1982). *Math for smarty pants.* Covelo, CA: Little, Brown and Company.

Intermediate and Secondary Tasks (Intermediate-Grade 12)

Triangle Numbers[4]
Ten checkers can be arranged to form a triangle, as shown in the image below.

The triangle is made up of a series of lines of checkers that are placed one above the other. The longest line is at the bottom, and each of the other lines has one checker less than the line that is just below it. Because ten checkers can be arranged in a pattern like this, the number 10 is called a *triangle number*. If we remove the bottom line of four checkers, what's left t is another triangle made up of six checkers, so the number 6 is also a triangle number. If we then remove the line of three checkers, what's left is the triangle number 3. If we remove the line of two checkers, we find that the smallest triangle number is 1.

If we reverse the process, starting with 1, we can get all the triangle numbers in the order of their size. Thus, 1 is the first triangle number; 2 is the second; 6 is the third; and so on.

What is the fifth triangle number? the tenth? the hundredth? Find a shortcut (e.g., an expression) to find the nth triangle number.

Extend: Let's call the first triangle number T(1), the second triangle number T(2), and so on. Make and complete the following list:

$$T(1) + T(2) =$$
$$T(2) + T(3) =$$
$$T(3) + T(4) =$$
$$T(4) + T(5) =$$
$$\ldots$$

What do you notice? Can you explain this pattern?

Road Trip[5]
a) A car went up a hill at a speed of 90 km per hour, and returned downhill at a speed of 100 km per hour. What was the average speed for the trip?

b) A car traveled from Town A to Town B at an average speed of 50 km per hour. How f
c) ast would it have to come back to make the average speed 100 km per hour?

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[4] *Adapted from* Adler, I. (1957). *Magic house of numbers*. New York, NY: Signet.
[5] *Adapted from* Adler, I. (1957). *Magic house of numbers*. New York, NY: Signet.

Sweet 16[6]

Place the numbers from 1 to 16 in the grid below so that they obey the horizontal and vertical equations. The circles contain odd numbers and the squares contain even numbers.

Image source: Michael Polaski

Have a great problem to share?

Contribute to this column!
Contact us at thevariable@smts.ca.
Published problems will be credited.

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[6] Polaski, M. (2016). *Sweet 16: A new numbers puzzle*. Atglen, PA: Schiffer Publishing Ltd.

Alternate Angles

Alternate Angles is a bimonthly column on problems from multiple perspectives: various methods that could be used to solve them, insights we get from their solution, the new paths that they can lead us to once they have been solved, and how they can be used in the classroom.

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Tackling Tessellations

Shawn Godin

Welcome back, problem solvers. In the last issue, I left you with the following problem:

> A tessellation is a tiling of the plane using one or more geometric shapes with no overlaps or gaps. Dutch artist M.C. Escher has many well-known pieces that consist of tessellations. Since tessellations can involve all kinds of complex shapes, we will concentrate on easy ones: regular polygons. Find as many configurations as you can of regular polygons that tile the plane without leaving gaps and without overlapping. You may use one or more different shapes in your tessellation.

This problem is based on activities that I have done with my students in the past. It is also inspired by discussions with Peter Taylor of Queens University about his desire to create rich activities for the Math 9-12 project (discussed in the last column). In particular, we wanted to develop activities that would give students opportunities to strengthen their algebraic modelling skills, which we will use later in this column.

This is a nice problem because we can attack it on a number of levels. Let's start by using manipulatives. You can either create your own regular figures, or you can use some existing ones. If you have a set of attribute blocks (sometimes called pattern blocks), you will have equilateral triangles, squares, and regular hexagons to play with. These are the regular polygons in the set, because in each figure, all sides have equal lengths and all angles have equal measure. You will also have an isosceles trapezoid with three equal sides (half a hexagon) and two different rhombi. Here, we will only use the regular figures from the set of attribute blocks.

Using just these three regular shapes (triangle, quadrilateral, and hexagon) students are able to come up with many tessellations. They will quickly discover the only three tessellations that use a single shape. In Figure 1 are the 3 regular tessellations.
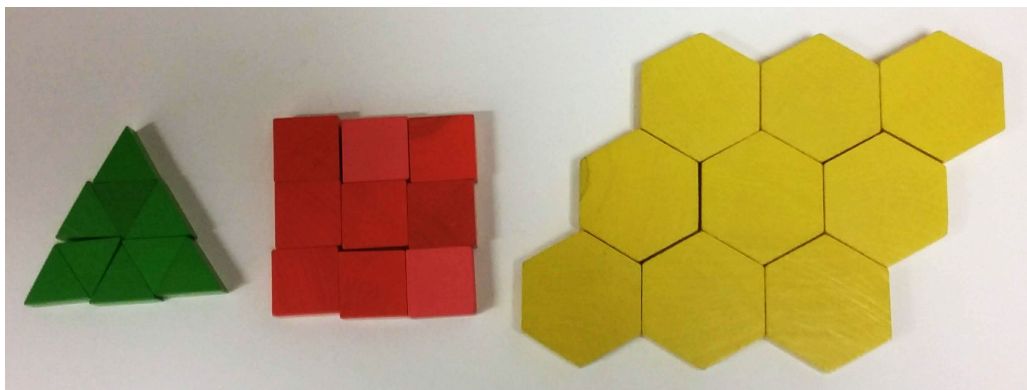
Figure 1: The three regular tessellations

It won't be long before students find other patterns. For example, they may see that six equilateral triangles can be put together to make a regular hexagon, as in Figure 2.
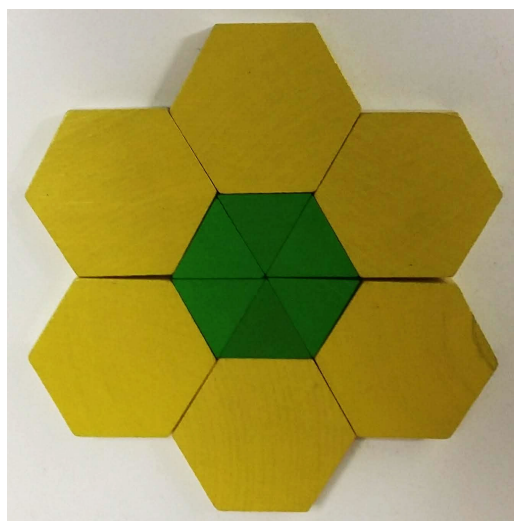
Figure 2: Triangle and hexagons

This gives us a strong relationship between the two tessellations that is tied into the geometry of the figures involved. There is a nice symmetry between triangles having 3 sides and 6 of them meeting at a vertex in a triangular tessellation, and hexagons having 6 sides and 3 of them meeting at a vertex in a hexagonal tessellation. We would say that the triangular tessellation is the *dual* of the hexagonal tessellation and vice versa.
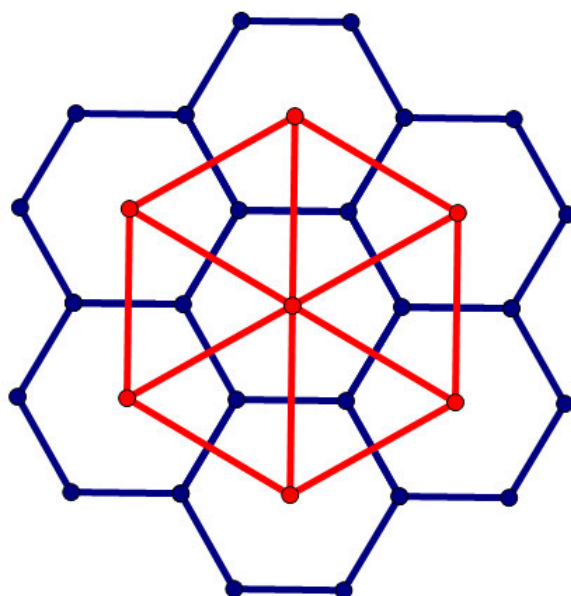
Figure 3: Dual tessellations

To create the dual of the hexagonal tessellation, place a point at the centre of each hexagon. When two hexagons share an edge, join the two centre points. When you go through this process you get a new tessellation where:

- polygons (hexagons) are getting replaced by points,
- line segments are being replaced by other line segments (the new and old segments intersect), and
- points are being replaced by polygons (triangles).

If you repeated the process with the new triangular tessellation, you would get its dual, the original hexagonal tessellation. What is the dual of the square tessellation?

When students explore further, they will discover more patterns involving multiple shapes. Figure 4 shows a tessellation where two hexagons and two triangles meet at each vertex, while in Figure 5 two squares, a triangle and a hexagon meet at each vertex.
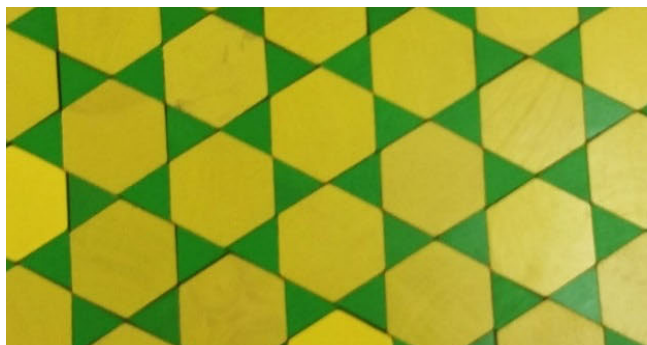
Figure 4: A tessellation with triangles and hexagons
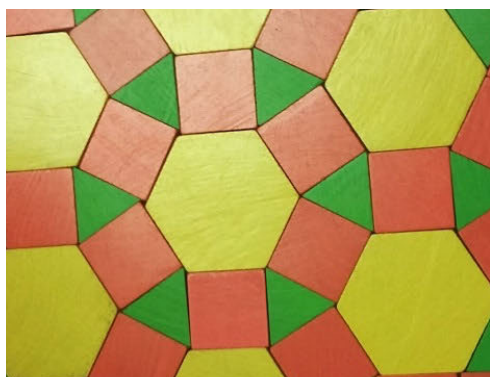
Figure 5: A tessellation with triangles, squares and hexagons

Students will enjoy looking for patterns and using their creativity to come up with new tessellations. I have used an exercise similar to this to my Grade 9 classes on a number of occasions.[1] When the full set of attribute blocks is used, many more patterns are possible. As students play, they may begin to wonder why certain combinations of shapes tessellate and others do not, leading to an investigation about the characteristics of angles found in tessellations.

Another great tool to attack this problem is dynamic geometry software, such as The Geometer's Sketchpad or Geogebra. We will use Geogebra[2] to investigate possible tessellations.

Using Geogebra's slider tool menu, select the slider tool as in Figure 6.
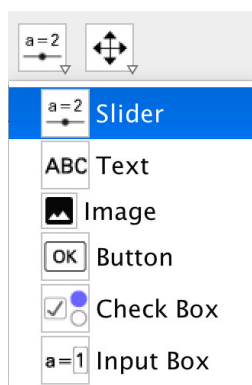
Figure 6: The Geogebra slider tool

You will be prompted with a dialog box as in Figure 7. The slider is going to be used to choose the number of sides a figure has, so you want the slider to be an integer and the minimum value to be 3.

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[1] You can check out a collection of my students' work at
https://photos.app.goo.gl/5v221Esh4jBCIiOp2.

[2] Available for free download at https://www.geogebra.org/download
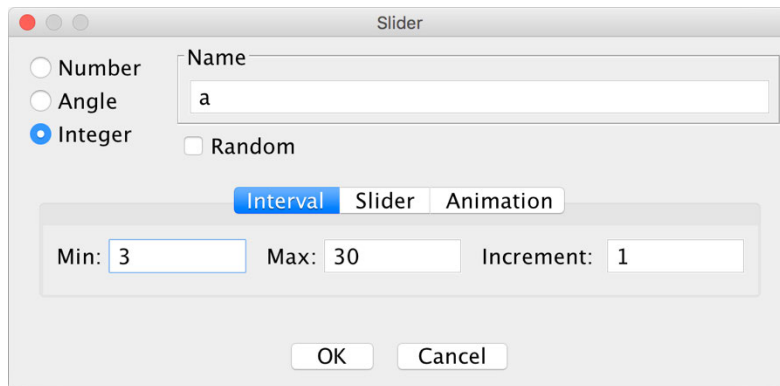
Figure 7: The slider dialog box

Next, we want to create a regular polygon where the slider controls the number of sides that it has. To do this, select the regular polygon tool from the polygon menu, as in Figure 8.
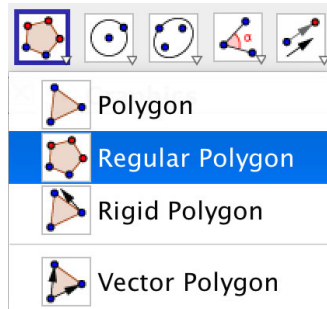
Figure 8: Selecting the regular polygon tool

Geogebra will now wait for you to create a line segment by indicating its two endpoints with mouse clicks. When you have done that, you will get a dialog box like the one in figure 9 asking you how many sides the figure has. Type the name of the slider, in my case "a", in the dialog box. Geogebra will create a regular figure with the number of sides indicated by your slider. As such, if you slide the slider, the number of sides on the figure changes accordingly.
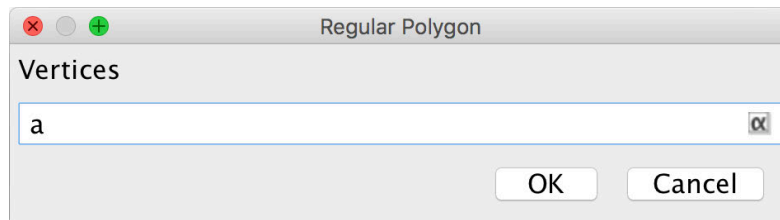
Figure 9: The regular polygon dialog box

We'll explore which combinations of three regular figures can meet together at a vertex without gap or overlap. If we repeat the whole process two more times (click on existing vertices when adding new polygons to ensure they move together), we can have a total of

three sliders controlling three figures that meet together at a common vertex, as in Figure 10.
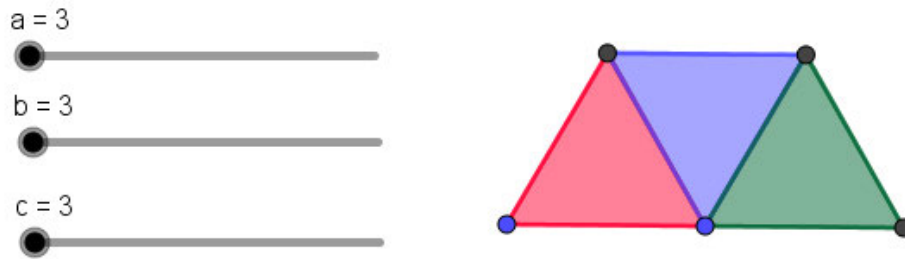
Figure 10: Three sliders and three figures (triangles)

We can play with the sliders to see what happens to the configuration for different regular figures. To start, we can keep of all the shapes the same, and thus change all the sliders to 4, then 5, then 6 producing the squares, pentagons, and hexagons shown in figure 11. Notice that when we get to hexagons, three fit together perfectly and could be used to reproduce the hexagonal tessellation we saw in figure 1. If you go beyond 6 sides for each of the figures, you will see the figures start to overlap, so we could stop looking for solutions of that type.
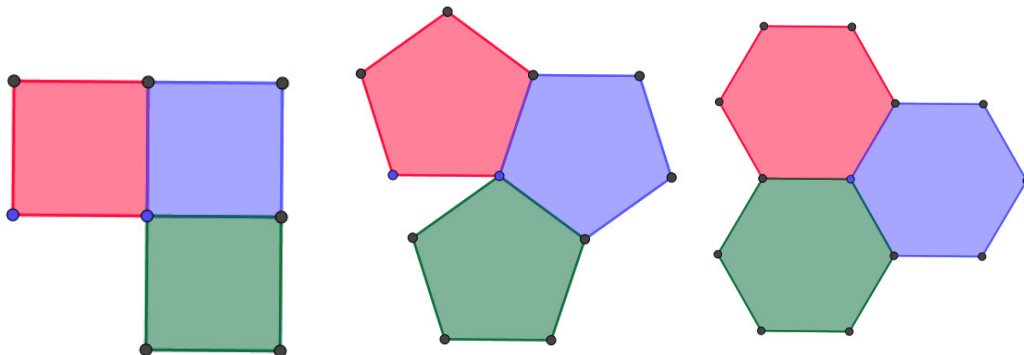
Figure 11: Three squares, pentagons, and hexagons at a point

Next, we could try looking for tessellations where two figures are the same and one is different. We know that two triangles meet to make and angle of 120° and two squares meet to make an angle of 180°. In both of these cases, we cannot find another figure to "make up" the rest of the 360° surrounding a vertex (why not?). Thus, we must start with two pentagons. We will find that three pentagons will not work, so if we play with one of the sliders, we will eventually find that two pentagons and a decagon seem to work, as shown in Figure 12.
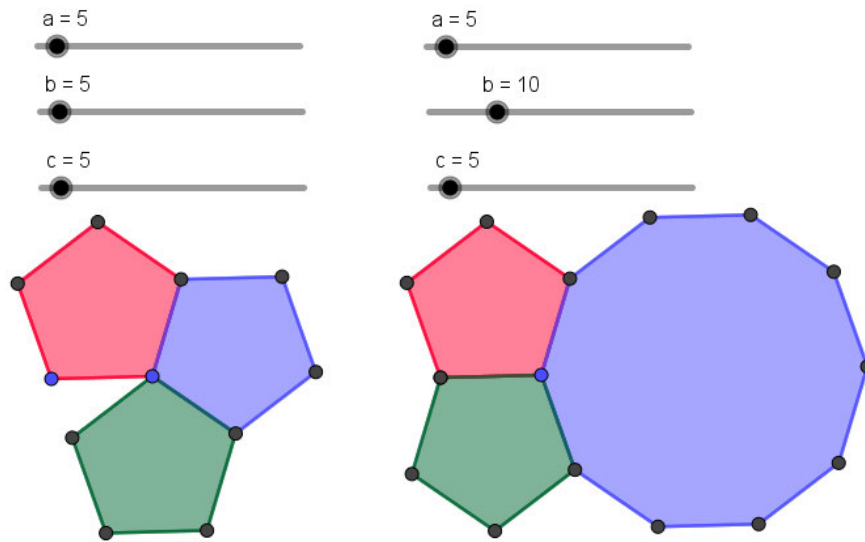
1.
Figure 12: Experimenting with two pentagons and another figure between them

Proceeding from here, we already know that three hexagons will work, so we can go on to two heptagons, then two octagons, and so on. When you have looked at all of the possibilities, you will find the other two solutions in Figure 13. It is interesting to note at this point that these are only *possibilities*. The three shapes fit together without leaving a gap, but this does not show that the pattern can be continued to form a tessellation. If you explore further, you will find that not all of the three examples found can be extended to a tessellation yet at least one does.
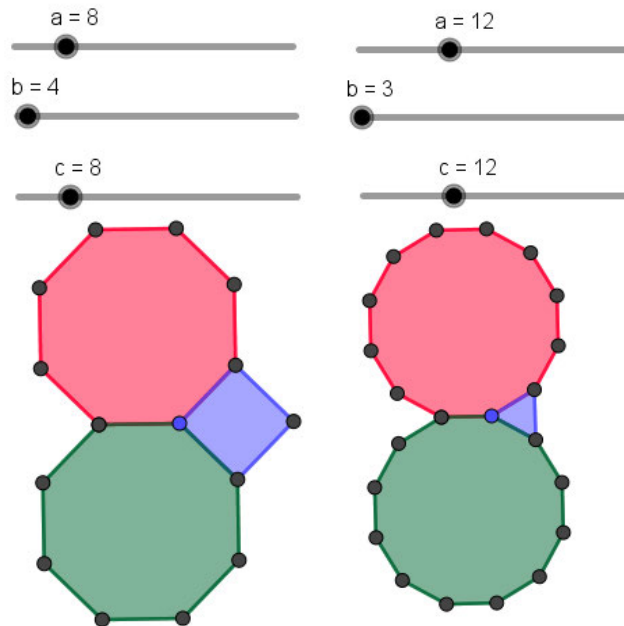
Figure 13: Two more three-figure possibilities

You can continue this line of investigation for four figures meeting at a point, then five figures meeting at a point. Once we reach six figures, we already know that six triangles meet to make the triangular tessellation and since the triangles have the smallest angles, we cannot replace them with any other regular polygon or there will be overlap.

It is interesting to note that once you get to four or five figures, the *order* in which the shapes are arranged is important. That is, you may find four shapes that meet at a point without gap or overlap, but depending on the order in which you arrange the shapes, the pattern may not continue. For example, Figure 5 shows a tessellation where each vertex is surrounded by a hexagon, a square, a triangle and another square. If you change this pattern to the only other possibility (why?)—hexagon, square, square, triangle—you will find that although you can continue to tile the plane, not all points contain the shapes in the same order. If you continue exploring groups of shapes that meet at a vertex without a gap, you should come across a collection of shapes that tessellate the plane in two different ways, depending on the order of the shapes around each point.

Now, let's explore attacking the problem symbolically. We will take it as known that the sum of the interior angles in a convex polygon with n sides is $(n-2)180°$, hence the size of each angle in a regular n-gon is

$$\frac{(n-2)180°}{n} = \left(1 - \frac{2}{n}\right)180°.$$

Using this, we can investigate groups of regular shapes that meet without gaps or overlaps algebraically.

Note that when a group of shapes meet at a point without gap or overlap, the sum of the angles that meet at that point must be 360°. Now, we will proceed in a similar fashion to our Geogebra investigation. That is, we will look at the possibilities for three shapes, then move on to four shapes and so on.

With three shapes, there are three possibilities: all three shapes are the same, two shapes are of one type and one of another, and all three shapes are distinct. We will start with the simplest case of three identical shapes, each with a sides. Then, the three angles around each point add to 360°, so we get

$$3\left(1 - \frac{2}{a}\right)180° = 360°$$

which, upon solving, gives us $a = 6$, or the hexagonal tessellation that we expected.

Moving on to the next case, where there are 2 shapes with a sides and one shape with b sides, things become a little more interesting. Again, the three angles around each point must add to 360°, yielding

$$2\left(1 - \frac{2}{a}\right)180° + \left(1 - \frac{2}{b}\right)180° = 360°$$

which, after a little algebraic manipulation, becomes

$$\frac{4}{a} + \frac{2}{b} = 1.$$

Finding common denominators and solving for *a,* we get

$$a = \frac{4b}{b-2}.$$

Keep in mind that we are searching for integer solutions to this equation. We can substitute in some values of *b* and see what happens. If we are doing this, a technological aid like a spreadsheet or Demos might be in order.

If we think about this logically, we can see that if *b* is odd, then so is *b* − 2, and the only factor they will share will be 1 (why?). Thus, unless *b* − 2 = 1, *b* − 2 will not be a factor of the numerator and we will not have a solution.

If *b* is even, the largest common factor of *b* and *b* − 2 is 2 (why?). What that means is that *b* − 2 must be a factor of 4 or 4 × 2 = 8, that is either 1, 2, 4, or 8. Substituting these in and solving for *a* and *b* yield the following configurations: dodecagon, triangle, dodecagon; octagon, square, octagon; hexagon, hexagon, hexagon; and pentagon, decagon, pentagon.

Notice that all of these cases are ones we discovered earlier with Geogebra. Also note that the equation also gave us the case where all the shapes were the same. What this means is that if we consider the equation

$$\left(1 - \frac{2}{a}\right)180° + \left(1 - \frac{2}{b}\right)180° + \left(1 - \frac{2}{c}\right)180° = 360°,$$

which simplifies to

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{2}$$

and find solutions, there will be cases where the three values are unique, cases where they are all the same, and cases where two the same and the other is different. That is, all cases are handled with this one equation.

Thus, we could have solved the case of three shapes using only this one equation. The downside to this strategy is that this equation is more complex to solve. By working with the simpler cases, we develop strategies that work, and since each new case includes the ones before, it gives us a check for our work. I will leave the final equation and the cases of four, five, and six shapes for you to play with. Remember, a solution is just a possible set of shapes that might be able to be used for a tessellation. We would also need to test to see if the collection of shapes (in some particular order) will actually tile the plane in a uniform way.

There are many more areas for investigation that I have not explored here, such as what happens when we replace a shape in a tessellation by a group of shapes that tile the original (think of six triangles replacing a hexagon, for example). We can also consider groups of

shapes that leave gaps. In this case, if the gap was closed by "gluing together" the edges on the outside of the gap, the resulting figure might be the vertex of a three-dimensional shape, a polyhedron. There are many more avenues where this exploration may go—have fun exploring them!

And now for your homework:

Explore the patterns suggested by

$$3^2 + 4^2 = 5^2$$
$$5^2 + 12^2 = 13^2$$
$$7^2 + 24^2 = 25^2$$
$$9^2 + 40^2 = 41^2$$

Until next time, happy problem solving!

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Shawn Godin teaches and is a department head at Cairine Wilson Secondary School in Orleans, Ontario. He strongly believes in the central role of problem solving in the mathematics classroom. He continues to be involved in mathematical activities: leading workshops, writing articles, working on local projects and helping create mathematics contests.

Spotlight *on the* Profession

In conversation with Dr. Ilana Horn

In this monthly column, we speak with a notable member of the mathematics education community about their work and their perspectives on the teaching and learning of mathematics. This month, we had the pleasure of speaking with Dr. Ilana Horn.

*I*lana Seidel Horn is Professor of Mathematics Education at Vanderbilt University. She uses interpretive methods to study secondary mathematics teachers' learning, seeking to improve education for students and supports for teachers, particularly in urban schools. Her current research project investigates mid-career mathematics teachers' learning in a rich professional development program.*

❖

First things first, thank you for taking the time for this conversation!

In your work, you often reference the notion of "ambitious forms of mathematics teaching," "ambitious instruction," or "ambitious practice" (e.g., Horn, 2012a, 2012b, 2013, 2015; Horn & Kane, 2015; Nolen, Horn, & Ward, 2015), often in connection to issues of equity. What are some teaching practices and beliefs that you view as ambitious, and how can they support the goal of successful outcomes for all students, regardless of racial, ethnic, linguistic, gender, and socioeconomic backgrounds?

Language is a huge challenge in educational research and practice. Every time we try to name something, it gets filled up with so many things that, as a concept, it starts to get squishy and meaningless. As a result, the term for mathematics instruction that prioritizes students' sense making has changed names many times in my almost 25-year career. "Ambitious" is just the latest term to try to stake out the territory of that kind of teaching. I got the term from Magdalene Lampert, who, along with Deborah Ball, did some important teaching experiments in the late 1980s and early 1990s on using rich forms of mathematical content as the basis for instruction that centered on sense making. When they were doing their work, they actually just called it "that kind of teaching" (TKOT). That doesn't really

work to communicate with the field, so I think that is how Magdalene finally landed on the name "ambitious," in part, to signal that it is not the kind of teaching we can expect teachers to have already encountered as students, nor the kind of teaching that schools are set up to support.

Can ambitious instruction really accommodate learners of all levels, including those referenced by your provocatively-titled article "Fast Kids, Slow Kids, Lazy Kids" (Horn, 2007)? Won't students who have generally moved quickly through content be bored or frustrated by having to work with students who tend to learn at a slower pace, and won't the latter group in turn feel discouraged and overwhelmed?

It fascinates me how many professional mathematicians bristle at the idea of rushing through mathematics. Mathematicians like Keith Devlin and Maryam Mirzakhani comment on the importance of taking your time and dwelling on mathematical ideas. In school, however, we organize things to value a quick pace and fast progress through the curriculum. As I describe in the paper you cite, the problem you describe is inevitable if we think math is a hierarchically arranged set of topics and skills that students learn with more or less facility that is primarily dependent on their capacity to calculate quickly and accurately. Then yes, absolutely, we will have that problem. However, if we think that math is a set of ideas—something like content and mathematical practices—and we think that there are multiple kinds of smartnesses that students can bring to rich mathematical questions, that aids the perceived "fast" kids by helping them understand the deeper connections in the content they are learning, and it aids the perceived "slow" kids by helping them use other kinds of mathematical capabilities to make sense of math. In other words, by enriching the mathematics students learn and get opportunities to do, and by broadening the range of authentically mathematical abilities we value in our classroom, we disrupt this linear notion of "fast" and "slow."

> "It fascinates me how many professional mathematicians bristle at the idea of rushing through mathematics."

On a recent research project I worked on, Kara Jackson at the University of Washington built on the ideas from *Fast Kids* and led the development of a measure called *Visions of Students' Mathematical Capabilities* (VSMC). This allowed our team to interview hundreds of teachers to try to ascertain how they thought about students' struggles in mathematics and how they acted in response to such struggles. We were then able to relate the measure to a lot of other things, like instructional quality and teacher growth. The VSMC measure turned out to relate to a lot of things, with more productive views correlating to better outcomes for students and teachers.

As you note in Horn (2015, p. 337), "ambitious forms of mathematics teaching are difficult to sustain, particularly when students' expectations are for other kinds of teaching." Further, you argue that such practices are even more difficult sustain in a culture of professional isolation and privacy, whereby teachers typically work alone in their classrooms and have little interaction with colleagues (Horn, 2013, 2015; Horn, Garner, Kane, & Brasel, 2017).

How does such a culture work against ambitious instruction?

As I have indicated, a hallmark of ambitious instruction is centering students' thinking. This inevitably increases the ambiguity of what teachers do. We can't count on the same ideas showing up in both sections of our geometry class, for instance, and we have to be ready to respond to what students do bring up—and, at the same time, we have to stay more or less with the same curriculum. Teaching is already full of uncertainty: Which kids will be in class today? What kind of mood will there be in the room? How will a school-wide event change their focus or energy? I could list a hundred such questions without thinking too much, acknowledging that the uncertainty is greater in some school communities than others. The added ambiguity adds a whole other layer of decisions for teachers to navigate.

> "A hallmark of ambitious instruction is centering students' thinking. This inevitably increases the ambiguity of what teachers do."

Teachers' success in managing the myriad decisions that go with centering students' thinking depends on their professional judgment. Even the brightest, most energetic, well-prepared and dedicated professional teacher has moments of deep uncertainty as they navigate these decisions. In typical school workplaces, where teachers work mostly in isolation, these moments of uncertainty can become a burden over time. In such situations, teachers' primary feedback about their "success" comes from their students. If the students dislike what the teachers are asking them to do—because it is too hard, too unfamiliar, too socially uncomfortable—that means that they are mostly getting negative feedback about their instruction. In contrast, if teachers work in a professionally supportive environment, they have additional resources to vet their judgments. This can be crucial when teachers are moving their own practice in the direction of ambitious instruction. A common make-or-break moment is when students complain that a teacher is not explaining the steps like their former math teachers. In this case, in a supportive environment, teachers have colleagues to consult with in such moments of uncertainty. Their colleagues may have tools or strategies to share. They may bring different strengths as teachers to model or bring to the particulars of the problem (e.g., talking to students outside of class to help get them "on board," locating engaging mathematical activities, writing thoughtful assessments). In this way, good colleagues provide another (non-kid) audience for teachers' work. They provide feedback on teachers' work that normalizes some common obstacle and that identifies success differently than students would.

> "Good colleagues provide another (non-kid) audience for teachers' work."

Teacher collaboration has frequently been identified in research as being concurrent with higher student outcomes, and as such, efforts to improve the teaching and learning of mathematics often include provisions for teachers to work together (Horn et al., 2017). Your own work has found that regular conversations with colleagues motivated teachers to persist with ambitious teaching practices (Horn, 2012a). At the same time, you have found that not all teacher collaboration is equally effective (e.g., Horn, 2015; Horn et al., 2017; Horn & Kane, 2015), writing that "collective work toward the goal of increased student learning is necessary but not sufficient for these kinds of outcomes" (Horn, 2013, p. 126).

What are some of the practices and/or beliefs that you have found to be in play in productive teacher teams that support teachers in sustaining ambitious instruction?

There are resources that teachers need to be successful in collective work towards ambitious instruction. First, they need access to rich mathematical curriculum, and, within the group,

there needs to be enough mathematical knowledge and mathematical knowledge for teaching to contribute to the collective thinking. Second, it is becoming clearer from things like the VSMC measure that teachers need to have an asset-orientation towards students and their learning to sustain and grow ambitious instruction. That is, they need to really have a strengths-based lens on individual and groups of students, focusing on what students are good at and how they are smart as much as they focus on what they do not yet know or understand. This is crucial, because as soon as deficit language seeps in, teachers hit roadblocks. This is why the VSMC measure focuses on how teachers think about struggling students. To illustrate how this works with an extreme example, if a teacher thinks that a student truly "can't learn," then there is nothing for the teacher to do. If, however, the teacher thinks that the same student "is really good at seeing patterns but doesn't know his times tables fluently," this opens up a range of possible instructional responses.

The way that teachers frame academic setbacks, as well as the kinds of mathematical competence that they value and support, are connected to larger beliefs about mathematics and mathematical ability. As you write in Horn (2007),

> If a category system explains students' success or failure by ascribing them varying degrees of ability and motivation, it delimits a range of reasonable pedagogical responses. Teachers can do little to change students' innate abilities—their best chance is to be engaging, perhaps overcoming low levels of motivation. Even if they manage to engage students, they may not be able to overcome the perceived deficiencies in their abilities to learn. (p. 74)

How malleable are these teacher-held beliefs, and what kinds of experiences can cause a shift towards a more complex view of mathematics and of mathematical competence?

This is, to me, the ten million dollar question. I don't think that we, as a field, have deeply engaged the phenomenon of teacher *learning*. There is a literature on teacher cognition, a literature on teacher beliefs, but teacher learning is less well understood. Clinically, I can tell you the primary way that I have seen teachers change their mindsets about students: When they observe a student whom they believed "couldn't" learn or engage meaningfully doing just that. I have known for some time that I can say "research shows" to a room full of teachers until I am blue in the face, and I will not change many minds. However, seeing a child that they know do something they did not think that child capable of doing—nothing seems to be more powerful in opening a teacher's mind. However, that moment alone does not guarantee transformation. Other resources have to be in place—professional development, a strong instructional coach, good colleagues—to make that moment meaningful in that teacher's development.

> "Seeing a child that they know do something they did not think that child capable of doing––nothing seems to be more powerful in opening a teacher's mind."

How malleable are the same beliefs in students who have generally not been attributed success during past years of schooling?

I think generally kids are easier than teachers. They are younger and still in the process of figuring out who they are. They are often willing to entertain the idea that they can be successful, despite prior struggles. In contrast, teachers work in a system that does not, on

the whole, support ambitious practice. If they open themselves up to more humane forms of instruction that reach more kids, they have to face the "oh no" moment, where they grieve the students that they may not have served as well. Not everybody can let their defenses down enough to have that moment. Students have a lot less to lose by believing they can learn.

Interviewed by Ilona Vashchyshyn

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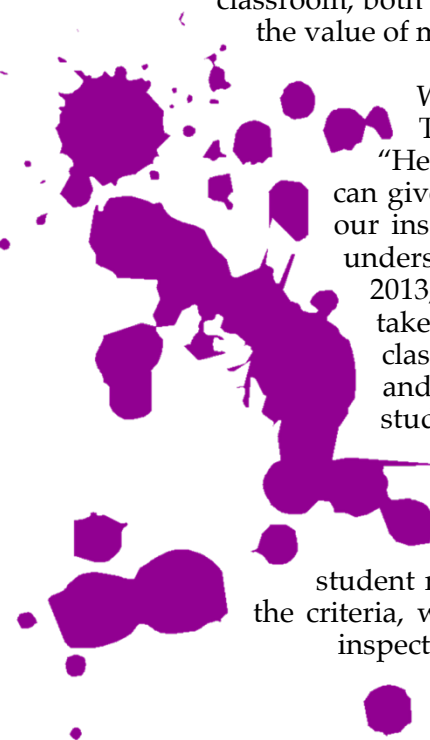
Examining Mistakes to Shift Student Thinking [1]

James C. Willingham, Jeremy F. Strayer, Angela T. Barlow, & Alyson E. Lischka

Middle-grades teachers and students can have different perspectives on the value of discussing students' mathematical mistakes, despite various classroom evidence that such discussions can help foster strong conceptual understanding (Boaler 2016). Some teachers consider student mistakes to be an opportunity to correct errors in individual student thinking. Others view the public inspection of mistakes as an opportunity for all students in the classroom to learn.

Although both of these perspectives take students' learning into account, our students often regard their own mistakes in a very personal manner. They see mistakes as flaws for which their teachers will judge them.

Because of the variety of perceptions regarding the value of mathematical mistakes, it is imperative that teachers consider how to leverage mistakes during classroom instruction. Are all mistakes created equal? How do we choose which mistakes are worthy of inspection? What exactly is the purpose of inspecting student mistakes? Is it simply to correct faulty answers? How should we use these mistakes during instruction? Reflecting on our teaching practice in light of these questions can lead to helpful insights. It is our hope that by considering the pedagogical quality of the mistakes examined in the classroom, both students and teachers will deepen their understanding of the value of mathematical mistakes for learning.

We begin with a statement by former National Council of Teachers of Mathematics (NCTM) President Linda Gojak: "Helping students to learn from their mathematical mistakes can give us insight into their misconceptions and, depending on our instructional reactions, can enable them to develop deeper understanding of the mathematics they are learning" (Gojak, 2013, para. 4). A number of classroom tools are available that take advantage of this powerful idea, including setting up classroom norms that value mistakes (Boaler, 2016); planning and selecting tasks to elicit mistakes (Bray, 2013); helping students focus on and discuss mistakes in meaningful ways (Pace & Ortiz, 2016); and assessing and designing responsive instruction based on student mistakes (Barlow et al., 2016). The purpose of this article is to add to this set of tools a list of criteria for determining *which* student mistakes are worthy of class examination. As we discuss the criteria, we offer insight into *why* certain mistakes are worthy of inspection and *how* teachers might leverage their examination to shift students' mathematical thinking forward within the context of a specific mathematics lesson. Our criteria are

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aimed at supporting the learning of *each and every* student, not just those who made the initial mistake.

Selecting Mistakes

Merely drawing attention to and correcting errors in calculation does not typically have a large pedagogical payoff for a deep learning of mathematics. Therefore, what kinds of mistakes might be worth inspecting so that students' mathematical thinking is moved forward in the learning process? Some mathematics education literature (Barlow et al., 2016; Kilpatrick, Swafford, & Findell, 2002; NCTM, 2014) provides guidance for deciding which student mathematical mistakes might be worthy of closer inspection. On the basis of this literature and our own experiences inspecting mistakes in classrooms characterized by students sharing their mathematical thinking and discussing their different solutions to non-routine, rich mathematics problems, we offer the following criteria for choosing inspection-worthy mistakes:

1. The mistake is closely aligned with the mathematical goals of the lesson.
2. The mistake provides powerful insight into students' conceptual understanding, fluency with procedures, or competence in selecting strategies for problem solving.
3. The mistake aligns well with the class's general progress toward solving the problem.
4. The mistake offers a viable answer that may be contrary to the class's accepted solution or solution strategies because of hidden assumptions about the problem (e.g., the student interpreted the problem differently than intended).

If a student mistake meets one or more of these criteria, it is probably a good candidate for whole-class inspection.

The remainder of this article describes how we applied these criteria to select mistakes for class inspection during a lesson with preservice teachers addressing a task designed for sixth- and seventh-grade students. The task focused on understanding ratios and percentages, and it elicited mistakes in the preservice teachers' work that were similar to those that might be expected from students at this grade level. For this lesson, we used a problem involving mixing paint, presented in the next section, as the central problem-solving task.

The Purple Paint Problem

The mathematical goal we sought to achieve with the Purple Paint problem was to have students track multiple part-whole relationships in a complex multistep problem and use these relationships appropriately to reason with ratios and percentages. The Purple Paint Problem follows.

> Katie wants to paint her bedroom a special shade of purple made up of equal amounts of pink and powder-blue paint. To make the pink paint for this mixture, she combines one part red paint with one part white paint. To make the powder-blue paint, she mixes three parts blue paint with one part white paint. Finally, to make the purple paint, she mixes equal parts of the pink and powder-blue paints.

If Katie needs two gallons of purple paint to finish her bedroom, how many quarts of blue, red, and white paint should she buy? What percentage of the purple paint comes from blue paint? What percentage comes from white paint?

Use diagrams, symbols, and words to justify that your answers are correct.

A sample of correct student work for this problem, indicating progress toward achieving our lesson goals, is included in Figure 1.

Fig. 1 This sample of correct student work for the Purple Paint problem met the lesson's expectations.

An equally important goal in our implementation of the Purple Paint problem was to allow each and every student in the classroom access to the rich mathematics embedded in the task. This access is required to produce mathematically meaningful mistakes that address the important concepts of the problem and support students in generating new mathematical understanding. For English language learners and other students who might struggle with reading or processing the task, we offer two suggestions:

1. In addition to posting the problem in a place where all students can read it, plan to read the problem aloud, physically demonstrate the actions involved in the problem, and offer appropriate translations. To ensure that students understand the intent of the task, ask them to revoice the problem while working in pairs, and then select a student to explain to the whole class what he or she thinks the problem is asking.

2. Consider the technique of Delaying the Questions (Barlow et al., 2017) to provide students an opportunity to make sense of the problem's underlying relationships before introducing the specific questions that they will be addressing. The Purple Paint problem is ideal for this technique because it contains a problem stem that introduces the relationships between each of the paint mixtures prior to the problem's questions. When students are allowed time to consider the ratio relationships ingrained in the problem stem, they will be much better prepared to apply their understanding of these relationships to the remainder of the problem.

Selecting Conceptual Mistakes Aligned with the Mathematical Goal of the Lesson
After introducing the problem, the teacher, who is also the lead author of this article, asked students to take a minute or so to think privately about how they might solve it before moving into groups of four to negotiate a solution. As students worked, the teacher circulated among them, observed their approaches, asked advancing questions, and considered which mistakes might produce an impactful whole-group discussion. The first mistake selected for whole-class inspection (see Figure 2) was chosen because it aligned with criteria 1 and 2. Specifically, the work focused on the central ideas of the lesson goal and contained a conceptual misunderstanding of the part-to-part relationships within the
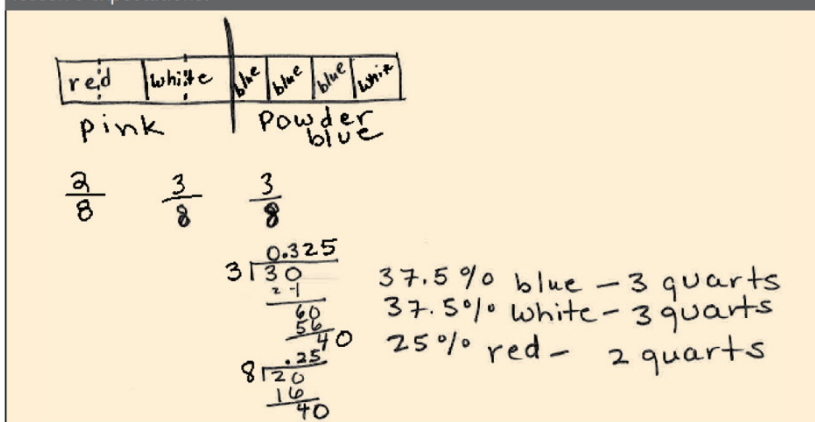
Fig. 2 This mistake demonstrated an error in conceptual understanding.

$$\boxed{R} + \boxed{W} = \text{pink}$$

$$\boxed{B}\ \boxed{B}\ \boxed{B} + \boxed{W} = \text{powder blue}$$

$$\boxed{R} + \boxed{W}\boxed{W} + \boxed{B}\ \boxed{B}\ \boxed{B} = \text{purple}$$

1 quart red
2 quart white } 1 gallon
3 quart blue } purple
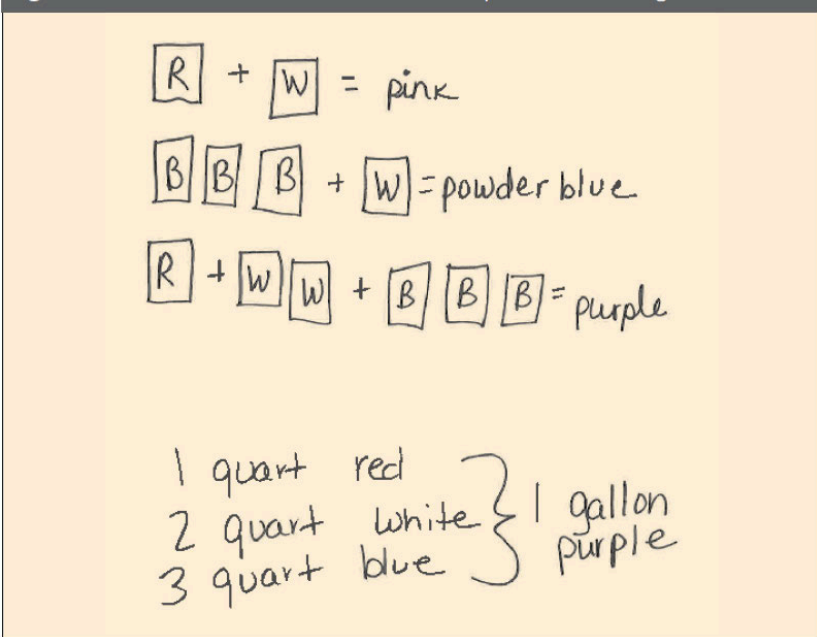
larger whole in the problem. Although the students who produced this work accurately represented the relationships of equal-sized parts in the smaller wholes (the mixtures of pink and powder-blue paints), they did not take into account how this relationship impacted the composition of the larger whole (the mixture of purple paint).

As groups of students began to complete the task, the teacher collected the sample in Figure 2 and displayed it via the document camera to be inspected by the class. Small groups were asked to compare the work with their own findings, focusing on how they had represented each of the mixtures in the problem. As students discussed this representation, several important ideas emerged and were shared with the class. One group agreed that the representations for the pink and powder-blue paints were correct by themselves but suggested that they had to be adjusted to show equal amounts of these mixtures in the purple mixture. A second group added that this would require either doubling the number of units used in the pink paint or halving the number used in the blue. Another group observed that the units in the representation were quarts of paint, and because the final amount of paint called for in the problem was equivalent to eight quarts, only doubling the number of units in the pink paint would actually give this amount. Through this process, students were able to shift their thinking to understand more deeply the idea of scaling a quantity based on its part-to-part relationship, a critical idea in solving problems with ratio reasoning.

Selecting Fluency Mistakes Aligned with Class Progress and Lesson Goals
The teacher selected the second mistake for inspection (see Figure 3) because it aligned with criteria 1, 2, and 3. This work contained an error in procedural fluency related to the conceptual error in Figure 2. Although the students who produced this work showed evidence of reasoning correctly on the first part of the problem, they calculated percentages for the subparts and assumed that those transferred directly to the larger whole. This procedural mistake was not only in line with the lesson's goal but also aligned well with the class's general progress toward solving the problem. In addition, this particular mistake was pervasive throughout the work of several groups.

The teacher directed students to consider this work in their small groups and then report on what they noticed. The resulting whole-class discussion focused on the meaning of *percentage*. Several students questioned the validity of their peers' claims that the white paint comprised 75 percent of all the paint when the work could also be interpreted as showing that the white paint made up 3 out of 8 parts of the whole. Other students noted

that it was unreasonable to say that red paint comprised 50 percent of the paint and at the same time claim that white paint made up 75 percent of the paint because this sum was greater than 100 percent without accounting for any of the blue paint.

The benefit of selecting a mistake based on criterion 3 was evident when students reflected on how the percentage error related to the previous discussion of the size of the parts compared with the whole (see the discussion of Figure 2). Indeed, students were able to move their thinking forward as they talked about the percentages, when referring to different wholes in the contexts of the pink paint, powder-blue paint, and purple paint.

Fig. 3 This work demonstrated errors in procedural fluency based on a previously examined conceptual error.
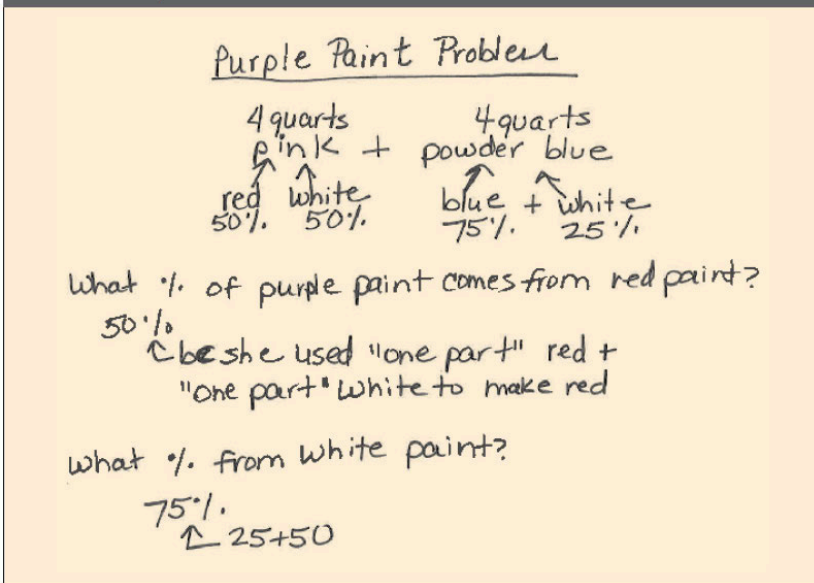
Selecting Mistakes that are Potentially Contrary to an Accepted Solution
Sometimes after a class has come to some conclusions about the solution to a problem, it can be beneficial to challenge their thinking. It was for this purpose that the teacher selected a final sample of student work (see Figure 4) on the Purple Paint problem according to criteria 1, 2, and 4. The students who produced this work showed a solution that contained dramatically different percentages from those that had been reported by most of the class. It appears that these students used reasoning similar to that of the students who presented the work in Figure 2 on the ratio portion of the problem, believing the purple paint mixture to comprise one part red paint, two parts white paint, and three parts blue paint. Although this reasoning was incorrect, the process they used to determine the percentages of the purple paint mixture

Fig. 4 In this sample of student work, percentages were correctly determined based on a previous part-to-whole error.
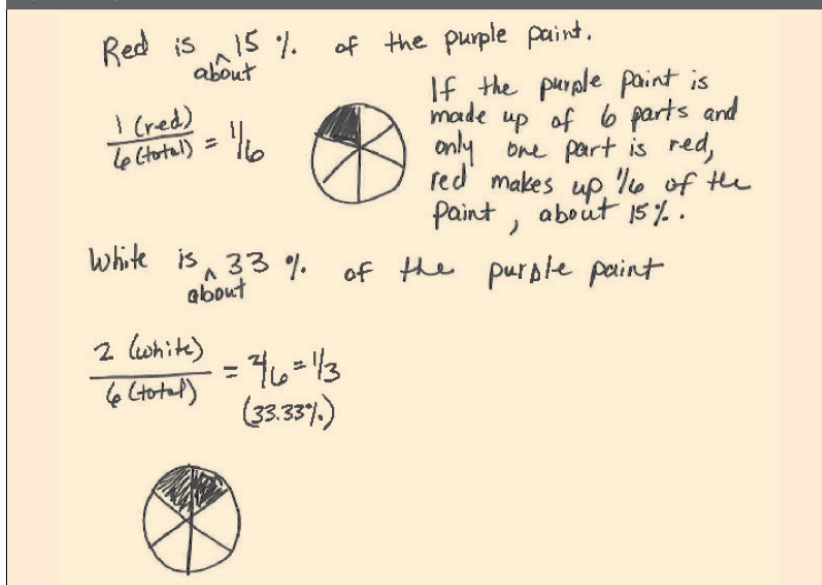
was appropriate. These students eventually resolved this error on their own, but the teacher used the Get the Goof strategy described by Pace and Ortiz (2016) to see whether other students could explain the mistake. Resolving the conflict that this mistake created provided an opportunity for the class to solidify their conceptual understanding according to the mathematical goals of the lesson.

Now that we have considered some examples of inspection-worthy mistakes, we find it helpful to distinguish them from those that are not inspection-worthy mistakes. We close the article with a brief discussion of mistakes that were made but were not chosen for class inspection during the Purple Paint problem.

Determining Which Types of Mistakes Will Not Be Inspected as a Class
If mistakes are minor, isolated cases and do not align well with the mathematical goals of the lesson, it can be difficult for a teacher to use them in a whole-class inspection with the goal of helping each and every student learn mathematics deeply. Often these mistakes are better addressed with brief one-on-one interactions with the student that do not detract from the lesson's primary mathematical goals. These kinds of mistakes occurred while students worked with the Purple Paint problem:

- Mistakes in calculation, such as errors in performing long division.
- Mistakes involving a missing piece of specific knowledge, for example, not knowing that there are four quarts in one gallon.
- Mistakes arising from a failure to read or represent the problem or its solution carefully (see Figure 5).
- Mistakes involving an inappropriate or incorrect application of a procedure (see Figure 6).

Fig. 5 Some mistakes arose from a failure to read and represent the problem or solution carefully.
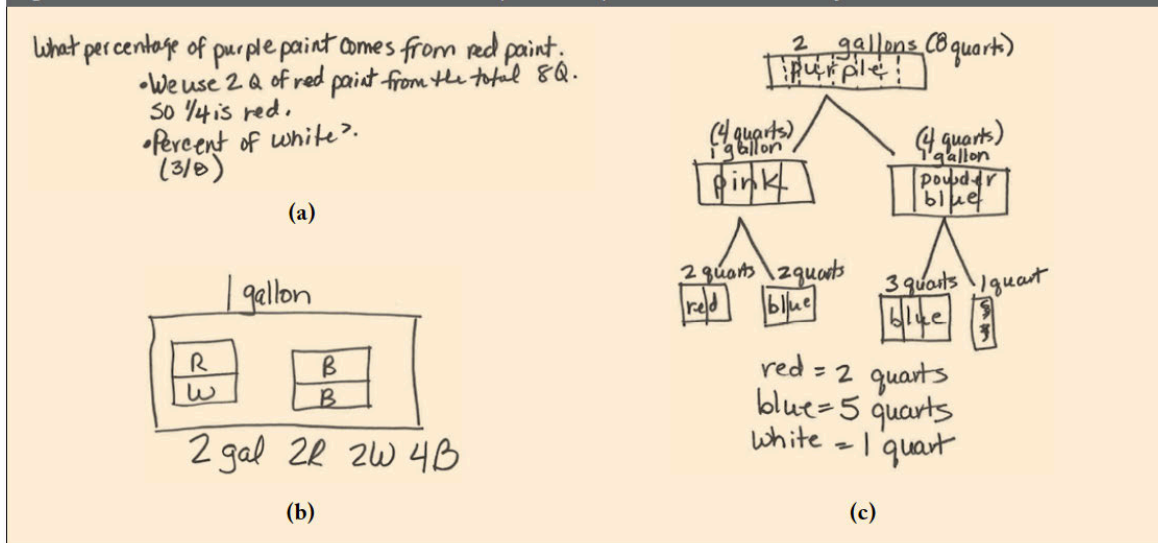
In Figure 5a, we see the work of a group of students who correctly determined that 3/8 of the purple paint was white but incorrectly stated that 3/8 was also the percentage of white paint. In Figure 5b, we see the work of students who treated the powder-blue paint as if it were pure blue paint and failed to include any white paint in their representation of the

powder-blue portion of the purple paint mixture. Finally, in Figure 5c, we see a representation in which the students incorrectly labeled two quarts of the pink paint blue instead of white. In these cases, the teacher can help students pay attention to their mistakes individually and move toward a solution that is aligned with the lesson goals by asking these one-on-one questions of each group: "What do you mean that the percentage of white paint is 3/8? Is there another way you can state this?" "Have you represented the white paint in the powder-blue paint? Why might this be important as you answer the problem?" and "I see that you have the pink paint made up of red and blue paint. Is that what you intended?"

Fig. 6 Some mistakes involved an inappropriate or incorrect application of procedures.

These personal interactions can be used to help shift thinking into mathematically productive areas and avoid the negative reactions that students sometimes experience because of mistakes of this nature.

Figure 6a displays work indicating that the students incorrectly carried out a procedure to determine the percentage value for a given fraction. Figure 6b displays work in which it is unclear how the students determined the percentages based on the number of parts identified. In each case, a teacher can help students individually move forward by asking such questions as "What do you mean by 400? Are you saying that two-eighths is 400 percent?" and "How do you know that the percentages you specified represent the number of parts of the whole you found? Do these percentages accurately represent what the problem states?"

Because the kinds of mistakes identified in this section hinder students from productively moving toward a solution that achieves the mathematical goals of the lesson, it is helpful for teachers to address them in brief one-to-one interactions with students, as described above. This enables students to spend their time wrestling with the big ideas aligned with the mathematical goals of the lesson. A prolonged whole-class inspection of these types of mistakes would probably not be a judicious use of class time because they do not meet the criteria identified for inspection-worthy mistakes. We are not claiming, however, that these mistakes are unimportant. Indeed, teachers may find these kinds of mistakes helpful for identifying the focus of future lessons, according to their students' needs.
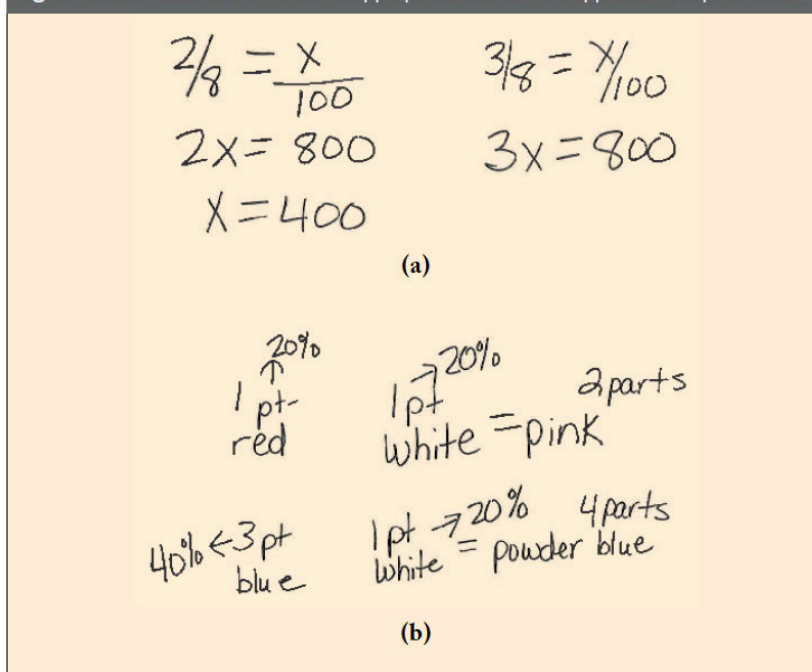
Inspecting Mistakes to Deepen Understanding

Conducting appropriate class inspections of mistakes in student work can create pedagogically powerful moments in the classroom. In this article, we present four criteria that teachers can use when deciding which mistakes to inspect in a whole-class setting so that students can shift their mathematical thinking and achieve deep mathematical understanding. By focusing on mistakes that meet these criteria, teachers can move the focus away from the fact that a mistake was made and toward the reasons why the mistake is mathematically meaningful for learning. It is our hope that as students gain expertise in examining meaningful mistakes, they will eventually regard this skill as one of the most important mathematical tools they have at their disposal when solving problems.

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What Makes a Function, Function?

Bryan Penfound

A quick image search using Google and the word *function* returns back exactly what I would expect from the average textbook: machine and arrow diagrams.
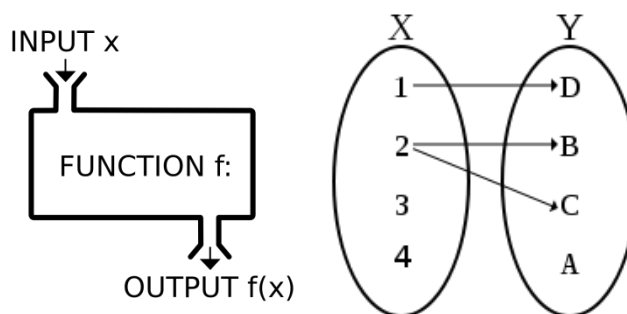

Figure 1: A machine diagram (left) and arrow diagram (right), both retrieved from Wikipedia.

While these may be simple ways to help illustrate the idea of a function via written medium, I was searching for more concrete examples to use during my college-level calculus class. Specifically, my goals were to help students understand the various types of relations and functions (one-to-one, many-to-one, and non-functions) and the two parts of the definition of a function in the context of non-algebraic examples. Admittedly, as a mathematics teacher, I often overlook the importance of concrete examples. However, just because I already know how to tackle questions using the algebraic structure of functions does not mean that my students are comfortable doing so.

I found a few problems from the instructor's guide for *Precalculus* (5th ed., Stewart, Redlin & Watson) that aligned with my objectives for the lesson: they featured concrete examples, dealing with relationships between objects like chairs, people, eye colour, and birthdays, but also connected well to my main goals of helping students see different types of relations and functions, and working with the two parts of the definition of a function. Using these problems, I created a Desmos activity[1] for the lesson.

We began the lesson by considering the following question:

> Do you think the following rule is a function? Why or why not?
>
> Domain: All people in this room
> Codomain: All chairs in this room
>
> *f(person) = his or her chair*

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[1] Available at https://teacher.desmos.com/activitybuilder/custom/59ae10b4cedfb95f83ab15be. I use Desmos extensively in my calculus class, in part because the activity builder allows students to submit responses in confidentiality and allows me to analyze responses for understanding.

A typical textbook response might be as follows:

> A function is a rule between a set of inputs and a set of outputs where each input gets mapped to only one output. Therefore, if each person in the room is matched uniquely to a chair in the room, the rule is a function.

However, before the question of whether the rule was a function could be addressed, a review of other concepts embedded in the question was warranted. Specifically, after a few minutes of pondering the question, the students indicated that they were curious about what a *codomain*[2] was, to which I replied, "You were probably expecting something else, right?" They told me that they were expecting the *range*. Accordingly, I asked them what the range of this rule was. After a bit of silence, I grabbed an empty chair and proceeded to alternate sitting down and standing up, chanting "In the range, not in the range; in the range, not in the range." My students enjoyed this and seemed to catch on, so I connected this concept back to the function $f(x) = x^2$, noting that for this function, too, the range is a subset of the codomain (specifically, the range is zero and all positive real numbers, yet the codomain is the entire y-axis).

We could now return to the original question posed. Some interesting student responses to the question of whether the given rule is a function appear below, with my comments following immediately.[3]

> Euphemia: "Yes, because if you input a person into the function, the result will be the person in his or her chair."

It looks like the basic idea of a function is here. I followed up this response by asking the student, "What happens if someone is standing? Do you think it is still a function?"

> Maryam: "You bet it is, the input and output is 1:1."

Very nice observation and language. However, I believe this student feels that every person in the room must be assigned to a specific chair, and hasn't considered the possibility of someone standing up in class.

> Benjamin: "I think that it is a function because the number of chairs in the room can be found when we find out the number of people in the room. Example: *f*(5 people) = 5 chairs."

Focusing on the possible numbers in the situation, Benjamin seems to have a misconception that the inputs and outputs of a function must be numbers. Perhaps this student has only seen functions with numerical inputs/outputs, in which case our concrete example presents a challenge. Over the course of the lesson, Benjamin would have the opportunity

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[2] The codomain is the set in which all outputs of a function are permitted to land, as opposed to the range, which is the set in which all outputs do land. For example, consider the function $f(x) = x^2$. The outputs are permitted to land anywhere on the y-axis, however, the outputs only fall on $y \geq 0$. In this case, then, we say that the codomain is the entire y-axis, and the range is $y \geq 0$.

[3] All names are pseudonyms, referring to famous mathematicians (e.g., Euphemia Lofton Hayes, Maryam Mirzakani, and Benjamin Banneker), generated in Desmos Activity Builder.

to work with a variety of concrete, non-numeric examples of functions, and the opportunity to revise his conception of functions.

Many Inputs to One Output

Before moving on to the next example, I asked my students for a definition of a function. No one was able to recall it from memory, so we did a quick search on the Internet and found a nice definition to work with: *a function is a rule between a set of inputs and a set of outputs where each input gets mapped to only one output.* Great! Now I had the opportunity to play around with the two important parts of the definition: (1) each input in the set gets mapped to an output, and (2) each input is mapped to only one output.

Next, we considered the following example.

Do you think the following rule is a function? Why or why not?

Domain: All people in this room
Codomain: The set of colors: blue, brown, gray, green, hazel, other

f(person) = his or her eye color

After giving students a few minutes to think, I first asked them what the range of this relation was, and we came up with {blue, brown, green}. With only three outputs, but seventeen students in the class, some were worried about multiple inputs being matched with the same output (several students having the same eye colour). One even mentioned that this would fail the vertical line test. "Interesting," I said. "Let's take a look at that claim." I drew a set of axes on the board and asked for the independent variable. After plotting a few persons on the x-axis, I proceeded to plot some eye colours along the y-axis. After plotting a few points, the students realized that this relation passed the vertical line test after all (see below). At this point, I brought up the algebraic example of $f(x) = x^2$ again, because it was similar to the eye colour example: each input has only one output, but multiple inputs can go to the same output (for example, $f(2) = f(-2) = 4$).
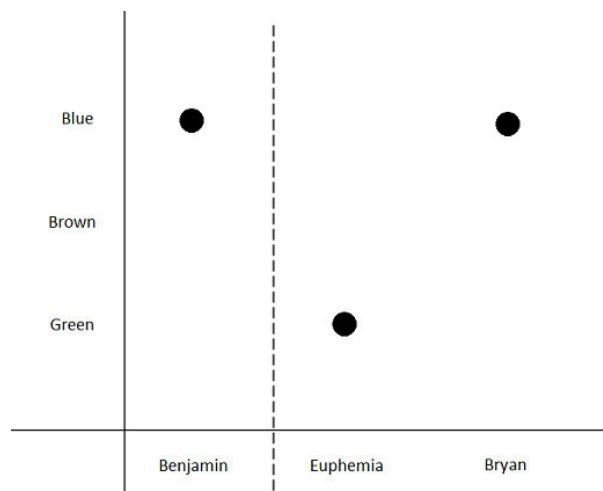

Figure 1: A plot of name versus eye colour. If we are able to shift the dashed line from left to right and it only passes through at most one point, the relationship passes the vertical line test.
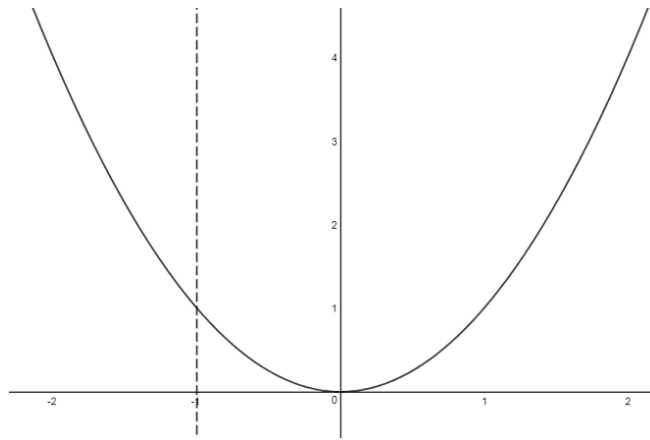
Figure 2: The graph of $f(x) = x^2$. One can see that if the dashed line moves from left to right, it passes through at most one point, so $f(x)$ passes the vertical line test and is a function.

I think my favourite comment during this example was "What if someone doesn't have eyes?" I'll let you ponder how to respond to this question! Some student responses to the original question of whether the given rule is a function are listed below.

> Joseph: "Yes, each input has 1 output."

> Al: "Yes, because each person only has one eye colour."

Both Joseph and Al are stating something very similar, with Al perhaps leaning more on the concrete example rather than on the definition. However, I believe that both show understanding of the concept of one-to-one-ness.

> Benjamin: "No because when you punch in the number of people in this room the answer won't give me a particular eye color. Ex: *f*(5) = *unknown eye color*."

Again, notice Benjamin's focus on a function requiring a number as input. However, interestingly, this student now is allowing for eye colour, a non-numerical attribute, to be an output. Reflecting more on this, I should have stopped the class at this point to focus on this response. Perhaps I could have asked, "What do we think this student is trying to say? Is it possible to write this in a different way?" This would have placed value on the student's response, and would have hopefully invited him to correct the response.

Every Input *Must* be Mapped
Our second-last example was the following:

> Do you think the following rule is a function? Why or why not?

> Domain: All people in this room
> Codomain: Days in September

f(person) = his or her September birthday

There was only one student in the room with a September birthday, so I wrote *f*(Blake) = Sept 10. After some discussion, I wrote *f*(Bryan) = ?, and most students realized that part (1) of the definition (every input gets mapped to an output) had been broken. With the relation not being a function, I mentioned that sometimes in mathematics, we restrict to a domain in order to make a relation a function. For instance, if we were to change the domain in our example above to {Blake}, *f* becomes a function (in fact, a one-to-one function). The algebraic example I connected to here was $f(x) = \sqrt{x}$: allowing all real numbers as input would make this a relation, but restricting the domain to $[0, \infty)$ makes it a function over the real numbers.

Some student responses to the original question of whether the given rule is a function are given below.

> Gotthold: "No as not everyone has a birthday in September so it would not be possible to have someone who does not get any output."

> Georg: "No, since the answers can only be September most of the inputs are undefined meaning they are not acted on, therefore no related output."

I like Gotthold's and Georg's use of the terms "input" and "output," and I especially like the use of the phrase "not acted on." They seem to understand one of the main aspects of the definition of a function: all inputs must get matched up with outputs. A good follow-up question for these students might be to ask specifically what the inputs and outputs are for this relation. Hopefully, this would clarify the response "answers can only be September," which I am looking for more specificity from.

> Benjamin: "No because there can be more people in the room than the amount of days in September. Also, not everyone will have a birthday in September."

It looks like Benjamin has picked up on the fact that a lack of output for some input is important, stating that "not everyone will have a birthday in September." The first sentence is also intriguing to me. Does he believe that once we match a person with a September birthday, the birthday gets crossed off a list (and that we eventually run out of days)? Does he understand that two people may share a September birthday and that this is OK, in the context of functions? I might ask this student what *f*(Blake) = Sept. 10 means, and if it is OK for *f*(Bryan) = Sept. 10 as well. Are we really going to "run out" of days in September? Benjamin's response reveals that he still has some work to do in understanding the concept of functions.

Each Input to *Only One* Output
Finally, since we hadn't yet broken part (2) of the definition of a function (each input gets mapped to only one output), I asked my students to create a relation using objects in the room such that (2) was not satisfied. Some responses are listed below.

> Al:
> > "*f*(person) = clothing items being worn
> > domain: all the people in this room
> > codomain: all possible clothing items"

I quite liked this response, which allowed for a discussion of how one input (one person) can get mapped to many outputs (jeans, shirt, socks, etc.), in which case the relation is not a function.

> Cathleen:
> "Domain: all the woodchucks in the world
> Range: all the wood chucked by those woodchucks
> f(woodchucks) = chucked wood"

My calculus students never fail to make me smile, so I had to include this response! Again, we can see that one input (a woodchuck) would be mapped to several outputs (all the wood that has been chucked). Note that the student has included the range, rather than the codomain. What do you think the codomain would be in this case[4]?

> Marjorie:
> "f(people) = favorite foods"

Another great response here, as some people may have multiple favourite foods. However, Marjorie didn't talk about the domain, codomain, or range of her relation, so I might follow up with her and ask what she thinks these sets should be.

Reflection

All in all, I was very happy with this activity, which I used for the purpose of review in my college-level course but that may also help students work towards outcome 30.1 of the Saskatchewan Calculus 30 curriculum ("Extend understanding of functions") or, with some adaptations, towards outcome 10.6 of the Foundations of Mathematics and Pre-Calculus 10 curriculum ("Expand and apply understanding of relations and function"). I believe that I was able to accomplish my original goals: to find some concrete, non-algebraic examples to help students understand relations and functions and to have students think about the two important parts of the definition of a function.

> "At the stage of concept formation, many students may find it beneficial to dial back abstract, symbolic examples in favor of examples that involve everyday objects or ideas."

Particularly at the stage of concept formation, many students may find it beneficial to dial back abstract, symbolic examples in favor of examples that involve everyday objects or ideas. As the work of my students shows, concrete examples can effectively develop vocabulary, address misconceptions (see some of Benjamin's responses), and extend mathematical concepts beyond the problem structures typically found in textbooks.

◈

Bryan is a college professor experimenting with open-resource assessments and cognitive science in the mathematics classroom. He is the creator of the Spring into Math event, which sees over 100 elementary students interact with playful mathematics at Okanagan College. You can find Bryan online at fullstackcalculus.com *and on Twitter via* @BryanPenfound.

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[4] A possible codomain for this example might be all of the wood in the world.

Intersections

In this column, you'll find information about upcoming math education-related workshops, conferences, and other events. Some events fill up fast, so don't delay signing up!

For more information about a particular event or to register, follow the link provided below the description. If you know about an event that should be on our list, please contact us at *thevariable@smts.ca*.

Within Saskatchewan

Structures for Differentiating Middle Years Mathematics
October 11, 2018
Lloydminster, SK
Presented by the Saskatchewan Professional Development Unit

We know that assessing where students are at in mathematics is essential, but what do we do when we know what they do not know? What do we do when they DO know? Understanding does not change unless there is an instructional response to what we know from that assessment. The question we ask ourselves is how we might respond to individual needs without having to create completely individualized mathematics programs in our classrooms. This workshop will include significant planning time for teams to get started in their own differentiation plans. It is also suited for educators who have already attended the Structures for Differentiation workshop.

More information at www.stf.sk.ca/professional-resources/events-calendar/structures-differentiating-middle-years-mathematics

Structures for Differentiating Elementary Years Mathematics
October 17, 2018
Lloydminster, SK
Presented by the Saskatchewan Professional Development Unit

We know through formative assessments that our elementary students are at different places in their understanding of mathematics, but how do we structure our classrooms to meet their individual needs? This workshop will provide the opportunity for participants

to design their classroom structure so that it allows children to move flexibly among large groups, small groups and individual instruction. By having a structure in place, teachers can create a differentiated learning experience without creating individualized learning programs for every child.

Head to www.stf.sk.ca/professional-resources/events-calendar/structures-differentiating-elementary-mathematics

Multi-Graded Mathematics
October 24, 2018
Moose Jaw, SK
Presented by the Saskatchewan Professional Development Unit

How do you address all of the needs within your combined grades mathematics classroom? By looking at themes across curricula, teachers can plan for diverse needs and address outcomes at two grade levels without having separate lesson plans. Curricular through lines and planning templates will be shared that are helpful for identifying how concepts grow over the grades, so that you can build a learning continuum within your instruction.

More information at www.stf.sk.ca/professional-resources/events-calendar/multi-graded-mathematics

Saskatchewan Understands Math (SUM) Conference
November 2-3, Saskatoon, SK
Presented by the SMTS

The Saskatchewan Understands Math (SUM) conference is for mathematics educators teaching in Grades K-12 and all levels of educational leadership interested in mathematics curriculum, instruction, number sense, problem-solving, culturally responsive teaching, and technology integration, and will bring together international and local facilitators to work in meaningful ways with participants in a variety of formats. This year, SUM is proud to welcome keynote speaker Lisa Lunney-Borden of St. Francis Xavier University.

More information at www.smts.ca/sum-conference/

Making Math Class Work
November 5, 2018
Tisdale, SK
Presented by the Saskatchewan Professional Development Unit

Math classrooms across Saskatchewan are increasingly complex and diverse. Meeting everyone's needs can be daunting, even with all of the instructional strategies and structures available to teachers. Number Talks, Guided Math, Rich Tasks, Problem Based Learning, Open Questions, High Yield Routines are just some of the strategies available to teachers, but where to start? Come work collaboratively to problem solve how to make math class work for you and your students.

More information at www.stf.sk.ca/professional-resources/events-calendar/making-math-class-work-2018

Beyond Saskatchewan

Geeks Unite 2.0: Math and Science Joint Conference
October 19-20, 2018
Edmonton, AB
Presented by the Mathematics and Science Councils of the Alberta Teachers' Association

Join the Mathematics and Science Councils of the Alberta Teachers' Association in celebrating their annual fall conference in Enoch, Alberta. This year's keynote speakers are Jo Boaler, Professor of Mathematics Education at Stanford University and faculty director of youcubed, and Jill Heinerth, a Canadian cave diver, underwater explorer, writer, photographer, film-maker and Fellow of The Royal Canadian Geographical Society.

More information at www.mathteachers.ab.ca/information-and-registration.html

57th NorthWest Mathematics Conference
October 18-20, 2018
Whistler, BC
Presented by the British Columbia Association of Mathematics Teachers

The BCAMT is proud to present the 2018 edition of the NorthWest Mathematics Conference in beautiful Whistler, BC! This year's keynote speakers are Tracy Zager, Annie Fetter, and Nat Banting. Featured speakers are Marian Small, Graham Fletcher, Michael Fenton, Christina Tondevold, Fawn Nguyen, Dan Finkel, and Chris Shore.

More information at bcamt.ca/nw2018/

Online Workshops

Education Week Math Webinars
Once a month, Education Weekly has a webinar focusing on math. They also host their previous webinars on this site. Previous webinars include Formative Assessment, Dynamic vs. Static Assessment, Productive Struggling, and Differentiation.

Past webinars: www.edweek.org/ew/webinars/math-webinars.html
Upcoming webinars: www.edweek.org/ew/marketplace/webinars/webinars.html

Did you know that the SMTS is a **National Council of Teachers of Mathematics Affiliate**? NCTM members enjoy discounts on resources and professional development opportunities, access to professional journals, and more. When registering for an NCTM membership, support the SMTS by noting your affiliation during registration.

NCTM | **AFFILIATE**
NATIONAL COUNCIL OF
TEACHERS OF MATHEMATICS

Tangents
Extracurricular Opportunities for K-12 Students

*T*his column highlights local and national extracurricular opportunities for K-12 students interested in mathematics, including collaborative, individual, online, and in-person challenges, contests, and camps. For dates, registration procedures and applications, and other information about the contests listed, please head to the contest websites, included below the descriptions. If we have missed an event that should be on our list, please contact us at *thevariable@smts.ca*.

If you are looking for contests available in other provinces, head to the Canadian Mathematical Society website (cms.math.ca/Competitions/othercanadian). The CMS also maintains a list of resources for students who are looking to build their problem-solving skills and succeed in competitions: see cms.math.ca/Competitions/problemsolving.

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Canadian Math Kangaroo Contest
Spring

The purpose of this competition is to introduce youngsters from Grade 1 to Grade 12 to math challenges in a fun and enjoyable way, thus inspiring their further interest and advancement in mathematics. The competition is held yearly in more than 40 Canadian cities. Students may choose to participate in English or in French.

More information at kangaroo.math.ca/index.php?lang=en

Canadian Team Mathematics Contest
Written in April
Presented by the Centre for Education in Mathematics and Computing (University of Waterloo)

The Canadian Team Mathematics Contest (CTMC) is a fun and challenging competition for teams of 6 secondary school students in any combination of grades. One teacher and groups of six students participate at their own school over 3 consecutive hours. The curriculum and level of difficulty of the questions will vary. Junior students will be able to make significant contributions but teams without any senior students may have difficulty completing all the problems.

More information at www.cemc.uwaterloo.ca/contests/ctmc.html

Caribou Mathematics Competition
Held six times throughout the school year

The Caribou Mathematics Competition is a worldwide online contest that is held six times throughout the school year. Each of these days, five contests are offered, one for each of the grade levels 3/4, 5/6, 7/8, 9/10 and 11/12 and each one in English, French and Persian. Available in English, French, and Persian.

More information at cariboutests.com

Euclid Mathematics Contest
Written in April
Presented by the Centre for Education in Mathematics and Computing (University of Waterloo)

The Euclid Mathematics Contest is an opportunity for students in their final year of secondary school and motivated students in lower grades to have fun and to develop their mathematical problem solving ability. Most of the problems are based on the mathematical curriculum up to and including the final year of secondary school. Most of the problems are based on curricula up to and including the final year of secondary school. Some content might require students to extend their knowledge and the best way to familiarize oneself with commonly appearing topics is to practice using past contests.

More information at www.cemc.uwaterloo.ca/contests/euclid.html

Fryer, Galois, and Hypatia Mathematics Contests
Written in April
Presented by the Centre for Education in Mathematics and Computing (University of Waterloo)

The Fryer, Galois and Hypatia Math Contests are an opportunity for students to write a full-solution contest. They are fun way to develop mathematical problem solving skills through a written mathematical activity. For students in Grades 9 (Fryer), 10 (Galois) and 11 (Hypatia). Questions are based on curriculum common to all Canadian provinces. Rather than testing content, most of the contest problems test logical thinking and mathematical problem solving.

More information at www.cemc.uwaterloo.ca/contests/fgh.html

Gauss Mathematics Contests
Written in May
Presented by the Centre for Education in Mathematics and Computing (University of Waterloo)

The Gauss Contests are an opportunity for students to have fun and to develop their mathematical problem solving ability. For all students in Grades 7 and 8 and interested students from lower grades. Questions are based on curriculum common to all Canadian provinces.

More information at www.cemc.uwaterloo.ca/contests/gauss.html

Opti-Math
Written in March
Presented by the Groupe des responsables en mathématique au secondaire

A French-language mathematics challenge for secondary students who would like to exercise and develop their problem-solving skills.

More information at www.optimath.ca/index.html

Pascal, Cayley, and Fermat Contests
Written in February
Presented by the Centre for Education in Mathematics and Computing (University of Waterloo)

The Pascal, Cayley and Fermat Contests are an opportunity for students to have fun and to develop their mathematical problem solving ability. For students in Grades 9 (Fryer), 10 (Galois) and 11 (Hypatia). Early questions require only concepts found in the curriculum common to all provinces. The last few questions are designed to test ingenuity and insight. Rather than testing content, most of the contest problems test logical thinking and mathematical problem solving.

More information at www.cemc.uwaterloo.ca/contests/pcf.html

The Virtual Mathematical Marathon
Supported by the Canadian National Science and Engineering Research Council

The virtual Mathematical Marathon has been developed by an international team of mathematicians, mathematics educators and computer science specialists with the help of the Canadian National Science and Engineering Research Council and its PromoScience program.

The main activity is a competition allowing students to enjoy solving challenging mathematical problems all year around. Students can join the game at any time and at no cost, simply creating an individual profile with an individual username and a password. Available in French.

More information at www8.umoncton.ca/umcm-mmv/index.php

Math Ed Matters by MatthewMaddux *is a bimonthly column telling slightly bent, untold, true stories of mathematics teaching and learning.*

Little Signs of Innumeracy

Egan J Chernoff
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First things first, let's give credit where credit is due. Roughly 30 years ago, mathematician John Allen Paulos coined the term *innumeracy* in his referentially titled book, *Innumeracy: Mathematical Illiteracy and its Consequences* (1988). Innumeracy, as the title suggests, and as he states in his interview with David Letterman[5], is the mathematical analog of illiteracy. A classic—and my favorite—example of innumeracy was the failure of a new burger released by A&W in the 1980s. The new third-pound A&W burger was less expensive than the McDonald's Quarter Pounder, and was even preferred in a taste test by consumers. And yet, contrary to expectation, the burger did not sell. Why? You guessed it, because three is smaller than four! Customers, as focus tests later revealed, misunderstood the value of 1/3 and believed that they were being overcharged.

Surely, we've come a long way since then... or have we? It seems to me that as time goes on, these signs of innumeracy only become more numerous and more commonplace. And so, in the spirit of Paulos' book, this column is dedicated to the little signs of innumeracy in today's world: that is, manifestations, and sometimes actual signs, of mathematical illiteracy.

Now, there are two possible scenarios that I must consider when it comes to these signs. *Scenario 1*: More and more signs of innumeracy are popping up around me because innumeracy itself is on the rise. *Scenario 2*: My attention, for whatever reason, has been drawn to these little signs of innumeracy and, as such, the signs seem to be popping up everywhere I go. In other words, it just *appears* that these signs are cropping up more often because I'm looking for them and, thus, I'm noticing them more than I have in the past. This phenomenon is sometimes referred to as the Baader-Meinhof phenomenon, or the

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[5] Available at https://youtu.be/HpOq0YlTOsM

frequency illusion. As someone who conducts research on heuristics and biases, I find myself in the quite the pickle here.

For what it's worth, my intuition points me to Scenario 1: that is, innumeracy is on the rise in the world in which we live. I believe this to be the case because of all the *new* signs I see today, along with the classic signs that Paulos identified in his book. These signs are cropping up everywhere, and I've been taking notes. And so, in an homage to Paulos' appearance on Letterman to discuss *Innumeracy*, here's my current Top Ten List of Little Signs of Innumeracy.

Number 10. Math problems stumping the internet.
Whether it's attempting to determine Cheryl's Birthday, find the right parking spot number, determine the age of the ship's captain, apply BEDMAS properly, or any of the other math problems designed for children in elementary school, they are stumping the internet. Stumping the internet, in this instance, means that the adults that are attempting to solve the problem are, by and large, getting the question wrong.

Number 9. Online banking.
I remember it like it was yesterday: sitting at the kitchen table, my Mom would get out a black pen, a red pen, some scrap paper, and her checkbook. She would then proceed to balance said checkbook. And it wasn't just my Mom: people everywhere would do something similar every month to determine how much money was spent and saved. Today, though, so long as you can remember the password to your online banking account, all that math is done for you and for your convenience. Maybe if we were still digging into the numbers in our bank accounts, we would be more cognizant of all that interest being carried on our credit cards… Out of sight, out of mind, I guess.

> "Maybe if we were still digging into the numbers in our bank accounts, we would be more cognizant of all that interest being carried on our credit cards."

Number 8. Mail Goggles.
I've used Gmail ever since it was released in 2004. Over the past 14 years, a number of experimental features have come and gone. Easily my favourite among these was a feature known as Mail Goggles, designed to keep you from sending late-night emails after partaking in too much beer, wine, or whatever your poison. Mail Goggles, whose name is reminiscent of a phenomenon known as Beer Goggles, served the same purpose as your friend who stops you from drunk-dialing your boss or an old acquaintance. Instead of a good friend, though, the little speed bump that kept you from sending the inebriated email was a series of math questions. Here's an example:

Mail Goggles ☒

It's that time of day. Gmail aims to help you in many ways. Are you sure you want to send this? Answer some simple math problems to verify.

Water and bed for you. Or try again.

395 + 93 =
198 + 108 =
56 / 7   =   53 seconds
185 + 161 =
178 - 93 =

(Send) (Cancel)

Rest assured: As the previous image shows, you weren't being asked to solve a differential equation. Nope, five questions involving varying levels of elementary arithmetic was the gatekeeper between you and sending those emails. I have it on good authority that the hardest level, level 5, was very good at preventing the sending of emails because you had to borrow to complete the subtraction. Nevertheless, it's a little telling that a multi-national tech company decided that answering some simple math problems was the hurdle that many could not overcome when they wanted to send an email while inebriated.

Number 7. The Machine.
While I did touch on The Machine in a previous column, there is no way it wasn't making this innumeracy Top Ten List. These days, when you're ready to the leave a pub or a restaurant, the server will make their way to your table, assess the situation, and, unless you're already holding cash in hand, proceed to ask you if you need The Machine. The Machine: a small, wireless miracle of technology that scans your credit card so you can pay for your meal. Where the innumeracy comes into play is when you are about to enter the tip. Gone are the days of mental arithmetic or scratching out some numbers on a napkin. These days, the server (typically) puts in the card and passes the terminal to you, which presents you with an array of options: 10%, 15%, 20%, or another percentage or dollar amount. Little left for you to do except to click.

> "I think it's cute that we're still debating whether calculators should be allowed in the math classroom or not. Meanwhile, as the debate continues, the outside world marches on."

Number 6. Photomath.
Personally, I think it's cute that we're still debating whether calculators should be allowed in the math classroom or not. Meanwhile, as the debate continues, the outside world marches on. And today's world includes Photomath. According to the description on the iOS App Store: "Simply point your camera toward a math problem and Photomath will magically show the result with a detailed step-by-step instructions." Let me say it again, because it bears repeating: not only does it solve the problem for you, it *shows* you, *step-by-step*, how to arrive at the solution. S.re, the app isn't perfect, but it's only going to get better. Maybe one day, Photomath technology will even be integrated with the tip-calculating machine so that I can actually see the steps involved in adding a 15% tip on a $100 bill.

Number 5. Birthdate bar entry.
As any young student learning mathematics will tell you, subtraction sucks. At least, it does when you have to borrow. As for when you have to borrow twice, don't get them started. Good news for all the cashiers who were forced into subtraction because they had to check the identification of patrons who wanted to buy cigarettes, lottery tickets, alcohol, or another substance or service restricted to legal adults: Instead of subtracting and (shudder!) perhaps having to borrow, signs, actual signs are now posted in bars and restaurants stating the date before which one must have been born to be of legal drinking age, eliminating any need for mental arithmetic. No subtraction, no problem.

Number 4. The Diff.
While I'm not a huge fan of regular season NBA, I do make sure to tune in to the NBA Finals every year. Those of you who have been following the NBA over the past five years will know that, each time, the Cleveland Cavaliers have played the Golden State Warriors. Without a doubt, the highlight of the past five NBA Finals has been the incident involving

the coach of the Golden State Warriors, Steve Kerr, making fun of "The Diff." The Diff, for those of you not already laughing, is a new addition to the scoreboard that hangs in (what is currently called) the Quicken Loans Arena in Cleveland. As you might suspect, The Diff calculates the difference, that is, the "diff," between Cleveland and the visiting team, saving fans and players from subtraction. Should Cleveland be winning, say, 74 to 72 for the moment, The Diff would show, you guessed it, +2. If Cleveland were down, say, 34 to 35 then The Diff would read -1. And, yes, if the score of the game was tied, The Diff would read 0. Again: No subtraction, no problem. [Please insert your own JR Smith joke/reference here.]

Number 3. The unit price is dead.
The unit price is dead. Well, to be a bit more specific, *calculating* the unit price is no longer a necessity. Walk into a modern grocery store or liquor store and, as shown in the photo below, the digital signage not only shows you the latest price for the item, but also the unit price. Whether it's per milliliter, gram, or whatever the appropriate unit, the unit price has been calculated for you, for your convenience, ahead of time. No more wondering whether you're getting a better deal if you buy the 6-pack or the 8-pack. A quick glance at the sign and you'll have all the information you need. As I said, the unit price is dead.[6]

Number 2. Percentage off sale signs.
Another literal sign of innumeracy found in stores these days is the Percent Off sign. In the event of a sale, the good people of various retail stores (e.g., Hudson's Bay, Wholesale Sports, and others) now often take it upon themselves to let you know, first, the original cost of the item, second, the percentage discount, and, third, the new cost, all on a neat sign beside the discount rack. Of all the signs in all the stores that I've come across, I've most appreciated the one that let me know that 10% off of the $100 dollar item that I wanted to purchase was now, after applying sale discount, going to cost me $90. What the sign ostensibly *didn't* show was whether you get a better deal if the tax is applied before the discount or after… "You see, it doesn't matter!" I explained to the cashier, which was met with, let's just say, consternation.

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[6] That said, maybe the unit price is only on life support. A stroll down the toilet paper aisle and your head may start spinning when it comes to single rolls, double rolls, triple rolls, one ply, two ply, three ply and all the other crazy calculations that are found on the package. Long live the unit price?!

Number 1. Incorrect percentage off sale signs.

Perhaps you're thinking, what could be worse than Percentage Off Sale Signs?! Well, check out the photo below. The sign reads: was $4.99; 5% OFF: $4.69.

My mental math, my calculator, and Photomath all agree… Not quite!

After all these signs, I am still unable to say, with certainty, whether there really has been an increase in signs of innumeracy or if I have just been noticing the signs more often because I've been on the lookout. And here's the real kicker: now that you've read my Top Ten List of signs of innumeracy, I am confident that you, too, will start to see them cropping up more often everywhere you go. Whether it's just another case of the frequency illusion or a real cause for concern, I hope we can agree: Innumeracy is everyone's problem.

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Ilona & Nat,
Editors