Escher Art
with Hexagons

Spotlight on the Profession:
In conversation with Dr. Richelle Marynowski

Illuminating Rectangle Border Challenges

Consumeracy: Consumer Numeracy

Saskatchewan Mathematics Teachers' Society

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Cover Image

SMTS Executive (2018-2020)

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The Variable welcomes a variety of submissions for consideration from all members of the mathematics education community, including classroom teachers, consultants, teacher educators, researchers, and students of all ages. Submitted material is assessed for interest, presentation, and suitability for the target audience.

Please submit articles by email to thevariable@smts.ca in Microsoft Word format. Authors should aim to limit submissions to 3000 words or less and include a photo and a short biographical statement of 75 words or less with their submission. Accepted contributions are subject to revision. Editors reserve the right to edit manuscripts for clarity, brevity, grammar, and style.

Message from the President

Happy November!

A slightly delayed release of *The Variable* has me able to say just how wonderful it was to kick off this winter season learning with all of you at this year's SUM conference. It never ceases to amaze me just how much more there always is to learn in this profession. I say this not necessarily referring to *volume* of learning (though, some days it certainly I feel that also) but more so in *nuance* of learning. I love that even though the core messages don't change in significant ways, our understanding of them shifts and deepens with time.

While mid-November is maybe not the best time to ask a group of educators to take some time to reflect, I would encourage you to find a quiet moment - even if it is in December - to reflect on pieces of your own practice and how they have shifted and deepened with time. Having the amazing opportunity to hear Dr. Lisa Lunney-Borden speak once again about the importance of the use of verbs in mathematics has really highlighted how we truly are lifelong learners. In 2014, Lisa's message of verbification and *doing* mathematics was simultaneously obvious and mind-blowing. It's been a fundamental shift in how I think about mathematics and how I work with other educators in doing mathematics. And yet! Hearing her speak on the same topic this year allowed me to really dig deep into analyzing my own use of language across the grades - something I think I will be able to reflect on continuously until I have the good fortune of learning with Lisa again.

It's easy to get lost on the mountain of things we don't know, don't know we don't know, or sure wish we knew when thinking of how best to meet the needs of all the learners in our room. However, we create spaces all the time for our students to reflect on their growth, to celebrate just how far they've come. Take some time and celebrate your own professional journey! What learning keeps popping back up for you in new and deeper ways?

Then, it is probably time to set one new goal as well. Be it one of the multitudes of ideas Mary Bourassa shared with us as to how she has begun *Planting the Seeds of Change* in her own journey, or something else you'd like to focus on - what will that one thing be this year? What deepening and nuance are you adding to your understandings? Who will you share this with in your community? As always, we'd love to be part of that community and hear about your thinking over on twitter or feature your reflection in *The Variable*. I am always grateful for these opportunities to come together as learners, and excited to see the directions each of us take our own learning. As always, stay mathy!

Michelle Naidu

My Favourite Lesson

Here at *The Variable*, we would like to amplify the work of Saskatchewan teachers through a new column titled "My Favorite Lesson." We invite you to share a favorite lesson that you have created or adapted for your students and that other teachers might adapt for their own classroom. There is no minimum length or strict limit on space. In addition to the lesson or task description, we suggest that including the following:

- Curriculum connections
- Description of the lesson or task
- Anticipated student action (strategies, misconceptions, examples of student work, etc.)
- Wrap-up, next steps

To submit a lesson or if you have any questions, please contact us at thevariable@smts.ca. We look forward to hearing from you!

Ilona & Nat,
Editors

Alternate Angles

Alternate Angles is a bimonthly column on problems from multiple perspectives: various methods that could be used to solve them, insights we get from their solution, the new paths that they can lead us to once they have been solved, and how they can be used in the classroom.

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Tracking Triplets

Shawn Godin

Welcome back, problem solvers. Last issue, I left you with the following problem[1]:

Explore the patterns suggested by

$$3^2 + 4^2 = 5^2$$
$$5^2 + 12^2 = 13^2$$
$$7^2 + 24^2 = 25^2$$
$$9^2 + 40^2 = 41^2.$$

One of the first things you might have noticed is that the pattern involves Pythagorean triples: that is, three positive integers (a, b, c) that satisfy the Pythagorean equation, $a^2 + b^2 = c^2$. The Pythagorean theorem tells us that a triangle whose side lengths form a Pythagorean triple is necessarily a right triangle. Perhaps you were then led to wonder: Which integers satisfy this relation?

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[1] This is Problem 4 from the book *Five Hundred Mathematical Challenges* by Edward J. Barbeau, Murray S. Klamkin, and William O. J. Moser, published by the Mathematical Association of America. This book is a great resource written by three mathematicians very involved in mathematical outreach and enrichment in Canada. It is a nice addition to your bookshelf and I would highly recommend it.

The method for generating all Pythagorean triples has been known for thousands of years. I recall, as a (much) younger person, when I first met the identities for squares of binomial sums and differences, I noticed that

$$(x + y)^2 - (x - y)^2 = 4xy = \left(2\sqrt{x}\sqrt{y}\right)^2$$

From there, I concluded that if x and y were both perfect squares, with $x > y$, then $a = 2\sqrt{x}\sqrt{y}$, $b = x - y$, and $c = x + y$ satisfy the Pythagorean equation. I remember filling a couple of pages of a notebook with examples verifying my discovery. At that point, I understood what the identity was telling me, but I still couldn't quite believe it! The general solution to the Pythagorean triple problem is usually written as: $a = 2mn$, $b = m^2 - n^2$, and $c = m^2 + n^2$, where m and n are positive integers. For now, though, we will only consider a subset of the solutions suggested by the pattern above.

Returning to the original problem, you should notice a few other things about the numbers in this pattern:

- the smallest numbers in each triple are consecutive odd numbers, starting at 3;
- the other two numbers in each equation are consecutive integers (4 and 5, 12 and 13, 24 and 25, and 40 and 41); and
- the consecutive integers add up to the square of the smaller number (i.e. $4 + 5 = 3^2, 12 + 13 = 5^2, 24 + 25 = 7^2, 40 + 41 = 9^2$).

So, how do we continue the original pattern? Can we describe, algebraically, each equation in the pattern?

One way to attack the general Pythagorean triple problem is by looking at some numerical examples to see if we find any patterns, such as the ones we found above. One way to do this is to either write a program to look for solutions, or to use a spreadsheet. As Figure 1 shows, I have entered possible values for a and b on the outer edges of the Excel spreadsheet (row 1 and column A).

B2			f_x	=SQRT(B$1*B$1 + $A2*$A2)				

	A	B	C	D	E	F	G	H	I
1		1	2	3	4	5	6	7	8
2	1	1.41421356	2.23606798	3.16227766	4.12310563	5.09901951	6.08276253	7.07106781	8.06225775
3	2	2.23606798	2.82842712	3.60555128	4.47213595	5.38516481	6.32455532	7.28010989	8.24621125
4	3	3.16227766	3.60555128	4.24264069	5	5.83095189	6.70820393	7.61577311	8.54400375
5	4	4.12310563	4.47213595	5	5.65685425	6.40312424	7.21110255	8.06225775	8.94427191
6	5	5.09901951	5.38516481	5.83095189	6.40312424	7.07106781	7.81024968	8.60232527	9.43398113
7	6	6.08276253	6.32455532	6.70820393	7.21110255	7.81024968	8.48528137	9.21954446	10
8	7	7.07106781	7.28010989	7.61577311	8.06225775	8.60232527	9.21954446	9.89949494	10.6301458
9	8	8.06225775	8.24621125	8.54400375	8.94427191	9.43398113	10	10.6301458	11.3137085
10	9	9.05538514	9.21954446	9.48683298	9.8488578	10.2956301	10.8166538	11.4017543	12.0415946
11	10	10.0498756	10.198039	10.4403065	10.7703296	11.1803399	11.6619038	12.2065556	12.8062485
12	11	11.045361	11.1803399	11.4017543	11.7046999	12.083046	12.5299641	13.0384048	13.6014705
13	12	12.0415946	12.1655251	12.3693169	12.6491106	13	13.4164079	13.892444	14.4222051

Figure 1: Pythagorean triples in Excel

In the diagram, you can see the formula in cell B2[2], corresponding to $c = \sqrt{a^2 + b^2}$. In the context of right triangles, the formula gives the length of the hypotenuse of a right triangle with legs of length a and b. We can now just look for the cells that contain whole numbers. In the table in Figure 1, we see four solutions:

$$3^2 + 4^2 = 5^2,$$
$$4^2 + 3^2 = 5^2,$$
$$8^2 + 6^2 = 10^2,$$
$$12^2 + 5^2 = 13^2.$$

The first two solutions are the same and the third corresponds to the first two solutions with all numbers doubled. The last solution is the second one in our original list.

If we continued with this spreadsheet, we would find more solutions that fit our original pattern, as well as some that do not. More importantly, we would find that solutions are few and far between. So, we need something to focus our search. Recall that earlier, we noticed that b and c in the Pythagorean triples in our pattern were consecutive integers. We can use this to narrow our search.
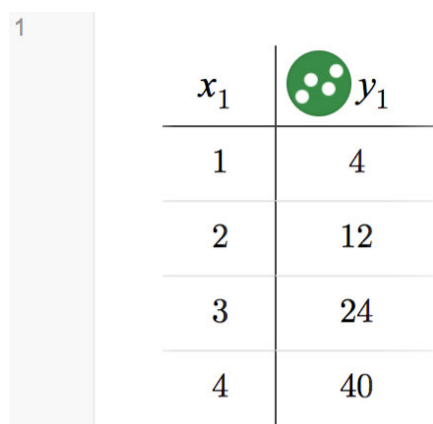
In Figure 2, the Excel search has been refined to the cases where b and c are consecutive integers. Now, we can look for consecutive numbers that produce our desired Pythagorean triples. In the diagram, you can see the formula in cell C2 used to produce the values of a from consecutive values of b and c. As we extend the table, we see that whole number values of a that satisfy the Pythagorean relation do indeed seem to be every odd number greater than 1.

C2		fx	=SQRT(B2*B2 - A2*A2)		
	A	B	C	D	E
1	b	c	a		
2		1	2 1.73205081		
3		2	3 2.23606798		
4		3	4 2.64575131		
5		4	5 3		
6		5	6 3.31662479		
7		6	7 3.60555128		
8		7	8 3.87298335		
9		8	9 4.12310563		
10		9	10 4.35889894		
11		10	11 4.58257569		
12		11	12 4.79583152		
13		12	13 5		
14					

Figure 2: Refining our search

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[2] Notice that there are several "$"s in the formula. The $ before the 1 in B$1 ensures that only the numbers in the first row (row 1) are used in the calculations, and the $ before the A in $A2 ensures that only the numbers in the first column (column A) are used in the calculations. This formula can now be copied to the other cells by dragging the small square to the right and down.

Now, let's try to investigate the structure of our solutions by looking closer at the values of b in these Pythagorean triples. If we go to Desmos, we can enter the table of values shown in Figure 3. In our table, the y values correspond to the b's and the x counts the first, second, third, … solution. Upon closer inspection, we see that while the first differences from row to row are not constant, the differences are increasing by the same amount (8, 12, 16…). Therefore, the second differences are constant, which means that this is a quadratic relation.

x_1	y_1
1	4
2	12
3	24
4	40

Figure 3: A table in Desmos

We can use regression to find the parabola of best fit. The quadratic regression for our table is shown in Figure 4.[3] The results indicate that the parabola of best fit has equation $y = 2x^2 + 2x$, while the coefficient of determination, $R^2 = 1$, indicates that the curve of best fit passes through all of the points. In other words, this is a perfect fit.
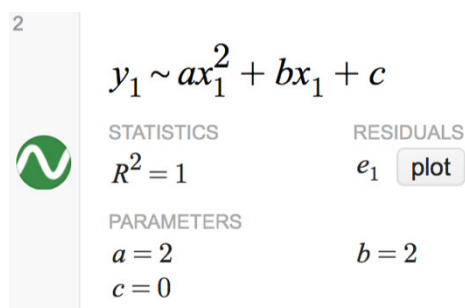

$$y_1 \sim ax_1^2 + bx_1 + c$$

STATISTICS   RESIDUALS
$R^2 = 1$ e_1 plot

PARAMETERS
$a = 2$ $b = 2$
$c = 0$

Figure 4: Predicting b with Desmos

Working backwards, if we let n be the solution number, that is, our x_1, then from the regression, $b = 2n^2 + 2n$; from our analysis of the table in Figure 2, which suggested that a takes on the values of all odd integers other than 1, $a = 2n + 1$; and finally, because b and c

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[3] The ~ indicates to Desmos that we want a regression, while the x_1 and y_1 in the expression tell Desmos where to get the data you want to fit. Indicating the coefficients as variables a, b, and c allows Desmos to calculate them for us.

are consecutive, $c = 2n^2 + 2n + 1$. A little bit of algebraic manipulation will verify that this will indeed yield a Pythagorean triple for each positive integer value of n.

That leads us to question: How could we have come up with this family of solutions without using technology? Since our pattern suggests that b and c are consecutive positive integers, let's set $b = m$ for some positive integer m and $c = m + 1$. Then, the Pythagorean equation gives us

$$a^2 + m^2 = (m + 1)^2,$$

which leads to

$$a^2 = 2m + 1.$$

Hence, a must be odd, just as our original pattern had suggested. If we set $a = 2k + 1$ for some positive integer k, after some algebraic manipulation we will arrive at the same solution that we discovered earlier.[4]

Alternately, if we rewrote the Pythagorean equation as

$$a^2 = c^2 - b^2$$

and factored, we would get

$$a^2 = (c - b)(c + b).$$

If b and c are consecutive, $c = b + 1$ and therefore $a^2 = (1)(c + b)$. which leads to the system of equations

$$c - b = 1$$
$$c + b = a^2.$$

The second equation also suggests that if a, b, and c are integers and b and c are consecutive, then $a^2 = 2b + 1$, meaning that a must be odd. Thus, we have found the same solution. The interesting thing about this method is that we can also apply it to cases when b and c differ by more than 1. If you play around with it, you may be able to solve the general problem. Have fun!

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[4] By the Pythagorean relation, we have that $a^2 = c^2 - b^2$. Substituting $c = b + 1$, we find that $b = \frac{1}{2}(a^2 - 1)$. Knowing that $a = 2k + 1$ for some positive integer k, we find that $b = 2k^2 + 2k$, and finally $c = 2k^2 + 2k + 1$.

And now for your homework.

A point P is drawn inside a particular square and is joined to the midpoints of each of the sides creating four quadrilaterals. The areas of three of the quadrilaterals are indicated in the diagram below.
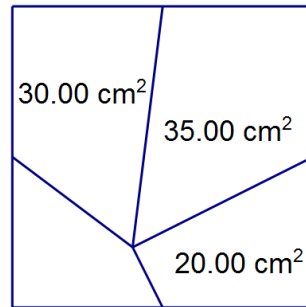
Determine the area of the fourth quadrilateral.

Until next time, happy problem solving!

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Shawn Godin teaches and is a department head at Cairine Wilson Secondary School in Orleans, Ontario. He strongly believes in the central role of problem solving in the mathematics classroom. He continues to be involved in mathematical activities: leading workshops, writing articles, working on local projects and helping create mathematics contests.

Problems to Ponder

Suppose you are given eight coins and you know that one is slightly heavier than the rest, which are of equal weight. You're also given a balancing scale that you use to compare the weight of the coins by putting some in one pan and some in the other. What's the minimum number of weighings you need to establish which coin is the heavier one?

What if you had started with 12 coins?

Adapted from plus.maths.org/content/weighing-balls

Call for Research Grant Applications

Notice of Intent 2019-20 Funding

The McDowell Foundation is inviting PreK-12 teachers and other educators to submit a Notice of Intent to begin the grant application process for research projects funded in the 2019-20 school year. Foundation grants provide funding, guidance, and release time to support recipients' project goals and activities.

Prior research experience is not necessary. The Notice of Intent and overall application process assists applicants in developing a meaningful and achievable research project which supports professional development through reflective practice. Grants totaling $85,000 are available.

The McDowell Foundation provides research grants to explore new and innovative ways to best meet the educational needs of students in Saskatchewan. A range of topics have been funded over the past 26 years and can be reviewed on the Foundation website, www.mcdowellfoundation.ca.

Notice of Intent Requirements

A completed Notice of Intent shall include the following:

1. **Applicant(s)**
    - Project leader's name, address, phone number and email.
    - Project team members' names, phone numbers and emails.
2. **Applicant Employer**
    - School, university or other.
3. **Describe What You Intend to Study and Why**
    - Provide a 100- to 200-word summary.
4. **Research Experience**
    - New, novice, or skilled researcher.

The Notice of Intent should be emailed to mcdowell@stf.sk.ca by the November 13, 2018, deadline. The Notice of Intent can be submitted orally if the research team prefers. Please contact the Foundation for further details.

Grant Application Process

1. Submit a Notice of Intent by November 13, 2018.
2. Applicants are invited to a grant proposal development workshop in December 2018.
3. Applicants develop a draft grant proposal for review by January 28, 2019.
4. Applicants may revise the draft grant proposal based on the feedback and advice received and shall submit a final grant proposal by April 22, 2019.

In conversation with Dr. Richelle Marynowski

In this monthly column, we speak with a notable member of the mathematics education community about their work and their perspectives on the teaching and learning of mathematics. This month, we had the pleasure of speaking with Dr. Richelle Marynowski.

*D*r. Richelle Marynowski is an Associate Professor in the Faculty of Education at the University of Lethbridge. Her areas of research are classroom assessment, large-scale assessment, teacher professional development, and mathematics teaching and learning. Dr. Marynowski was a high school mathematics teacher for 17 years before starting at the University of Lethbridge in 2011. She has taught in a variety of different contexts: K-12 rural school, outreach schools, alternative schools, and large urban schools. She is also currently a Teaching Fellow and a Tier II Board of Governors' Research Chair at the University of Lethbridge which are recognitions of her commitment to teaching and research.

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First things first, thank you for taking the time for this conversation!

Much of your recent work has centered around student assessment—in particular, how teachers use and relate to various types of assessment, including both classroom and high-stakes assessments. With regard to the former, you have discussed the affordances of formative assessment in the mathematics classroom in Marynowski 2014 and 2015. Although the phrase "formative assessment" (or "assessment for learning") is now common in curricula, such a catchall term is bound to be interpreted in different ways by different teachers.

What do you mean by formative assessment, and how does it contrast with—and complement—summative assessment? What are the advantages of formative assessment in the math classroom?

When I think about formative assessment, I have two different senses of what that looks like in a classroom: formative assessment as an informal event and as a formal event. As an informal event, formative assessment generally means to me all of the minute-to-minute

decisions that we make as teachers on a daily basis. We take in information that students give us about their learning and make a decision about how to proceed. Sometimes these decisions are so embedded in our teaching practice, we don't even know we are doing it. More formally, formative assessment can be a product, like completing an exit slip, that can give a teacher specific information about students' learning. I truly believe that formative assessment is embedded in teaching and cannot be separated from it.

Formative assessment is what tells us whether students are ready for a summative assessment. If we feel that our students are or aren't ready for the test, project, presentation, etc., we have used some sort of formative assessment to determine this. Formative assessment is continuous, while summative assessment is more formal and more of an event in a classroom. The two should supplement and be consistent with one another—if students perform unexpectedly on a summative task, or inconsistently with what the teacher knows from all of the formative evidence, then the teacher is cued to inquire more about the summative assessment to see if something was amiss.

> "When we pay attention to student work and to student conversations, we learn so much about what and how students know what they know."

In the math classroom, formative assessment is also key to helping build student knowledge. Without formative assessment, how do we know when students understand something or can do it? On the test? By then, it's too late! As teachers, when we pay attention to student work and to student conversations, we learn so much about what and how students know what they know that we couldn't by just looking at summative work.

Does formative assessment mean (even) more grading for the teacher?

Absolutely not! I would say it means less grading and more getting to know what students know—either informally or formally. Every time we ask students a question, pay attention to the answer, and respond with either some missing knowledge or respond to extend knowledge, we are using formative assessment.

In Marynowski (2015), some teachers who engaged in professional learning experiences centered on formative assessment realized, with some surprise, that formative assessment "doesn't have to be from me [the teacher]" (p. 9). Could you explain?

This really focuses on leveraging students as resources for each other. When students give each other feedback on their work, they are performing formative assessment. In setting up classroom spaces or structuring student practice, getting students to explain their work to each other and provide specific feedback on that work allows them to build their mathematical fluency as well. In *Embedded Formative Assessment,* Dylan Wiliam (2005) talks about activating students as instructional resources for each other as one of the five ways of integrating formative assessment into one's practice. (The other ways of embedding formative assessment into one's teaching practice discussed by William are making learning intentions clear and sharing success criteria with students; eliciting classroom discussions and tasks that bring forward evidence of learning; providing feedback to students that moves their learning forward; and activating students as owners of their own learning—that is, self-assessment. I have used these categories of strategies to illustrate to teachers how they currently use formative assessment in their teaching without perhaps

recognizing it in the first place; see Marynowski, Mombourquette, & Slomp, 2017, and Marynowski, 2015.)

I have seen many mathematics classrooms where students work quietly and individually, sharing their mathematical thinking only with the teacher. In classroom spaces where students are expected to talk to each other about their work and think through their work with each other, students have more opportunities to clarify their understanding and come to see how others understand something. Seating students in pairs, designing activities where they need to work together to complete the task, cooperative learning strategies, Kagan structures, row games, C3be4me (see three before me–meaning students must talk to at least three other students to get help before they come to the teacher for help, which means that the teacher should then have a group of 4 coming to see them instead of individuals)—any of these strategies, among others, get students sharing their thinking in formative ways. In communicating and working together, students can get new insights from each other that they can then take and integrate into their own sense of the ideas.

Is the goal of formative assessment to prepare students for summative assessment (e.g., a unit or final exam)? I wonder if you could speak to the tension between the two, particularly when high-stakes evaluations are involved. (As you explain in Marynowski 2016, in Alberta, students are required to take provincially developed examinations, called diploma examinations, for certain Grade 12 courses. In 2015, diploma examinations counted 50 percent toward students' final grades in these courses. Similar examinations are not currently mandated by the Saskatchewan Ministry of Education.)

The goal of formative assessment is to improve student understanding, which, in turn, will prepare students for whatever summative assessment they will be asked to complete. To me, formative assessment is "assessment in the service of learning," not a practice for high-stakes, or any, exams. The root of the word "assess" is the Latin word *assidere*, which means "to sit beside." So formative assessment is a way in which a teacher can sit beside a student to come to know what that student knows. Assessment is a process of coming to know what a student knows in many different ways.

> "To me, formative assessment is 'assessment in the service of learning,' not a practice for high-stakes, or any, exams."

The history of examination design and development is from psychology, which is a more clinical and objective way to look at assessment that does not always work in an educational setting. So there is absolutely a tension between government-mandated examinations and formative assessment—but these two types of assessment serve two very different purposes and come from different histories, so there should be tension. What frustrates me is that society prioritizes the perceived objectiveness of exams over the knowledge of teachers about what students know and can do.

Last year, you co-led a working group titled "Deep Understanding of School Mathematics" along with Peter Liljedahl and Sarah Dufour, at the 41st annual meeting of the Canadian Mathematics Education Study Group.

Although the purpose of the working group is to explore and further develop understanding of the topic, I wonder if you could offer your own (perhaps working) perspective on what it means to have a deep understanding of (school) mathematics.

This is such a great question and one that I have been trying to articulate for a while, and thus my perspectives keep changing—but I will give it a go! Having a deep understanding of school mathematics is having a sense about how the different pieces all fit together and work together to create mathematics as a discipline. I don't have to know *all* of mathematics to have a deep understanding, but I do need to have a sense that what I am learning is part of that larger whole and fits together, as well as a sense of the flexibility of how to work within the rules of mathematics and when to draw on what knowledge. With respect to flexibility, I envision this meaning that students are able to take their knowledge of mathematics and apply those ideas to a problem to solve it in a different way. Having a deep understanding of school mathematics means being able to connect what, to others, might seem to be very different concepts: for example, geometric transformations of shapes and transformations of functions. In school mathematics, those topics are taught separately and often years apart. A student who has a deep understanding can draw on what she knows about shape transformations and link it to new learning about function transformations. I equate this to having a "gut" sense or understanding about how math works, not just a "brain" understanding… if that makes any sense!

Relatedly, how might teachers teach for deep understanding within time and curriculum constraints, and what might assessment for and of deep understanding involve?

I think we have an obligation to help our students strive for deep understanding, which we can do if we stop seeing our curriculum as separate bits of knowledge that have to be 'covered' and try to find those deeper connections. For example: unit rates are connected to proportions, which are connected to enlargements and reductions of objects, which are also connected to slopes, which are connected to equivalent fractions… and I am not sure we help our students see those connections. Thus, before we can even try to help our students, we have to be open to seeing mathematics in a connected way and not let ourselves get drawn into splitting math into units of study. I imagine a curriculum that is built more on conceptual understandings rather than discrete topics, and that the 'big ideas' (Marian Small) or 'essential understandings' (Wiggins & McTighe) become the goal of the curriculum rather than bits of knowledge. If we pull ourselves out of the minute details of the curriculum (which are easy to identify) and try to see the bigger picture of what are the essential elements and really big concepts of what we are trying to accomplish in math education, then I think we can do it. Assessment then flows right along with that, so that instead of asking students to complete 15 questions on all of the different kinds of factoring questions we can think up, we ask them to describe their thought process as they engage in one factoring question. With respect to assessment, we have to think of less and slower, rather than more and faster.

> "Before we can even try to help our students, we have to be open to seeing mathematics in a connected way and not let ourselves get drawn into splitting math into units of study."

Interviewed by Ilona Vashchyshyn

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Wiliam, D. (2005). *Embedded formative assessment.* Bloomington, IN: Solution Tree Press.

Problems to Ponder
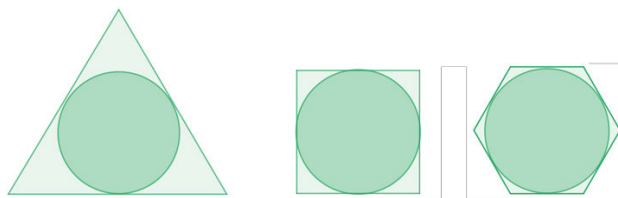
In the diagram above, various regular polygons, P, have been drawn with sides tangent to a circle, C. Show that for any regular polygon drawn in this way:

$$\frac{Area\ of\ P}{Perimeter\ of\ P} = \frac{Area\ of\ C}{Circumference\ of\ C}$$

Source: http://mei.org.uk/month-item-14

Illuminating Rectangle Border Challenges[5]

Jacqueline Coomes

On my way home from a conference a few years ago, I was browsing through books in an airport bookstore and came across *Number Freak: From 1–200, The Hidden Language of Numbers Revealed*, by Derrick Niederman (2009). I purchased it and became engrossed in solving problems throughout the flight home. After solving one problem in the book (see Figure 1), I was left with some vexing questions. How could I have known that there were only two rectangles with this property if the author had not divulged it in the question, and how could I have anticipated the other solution? Exploring answers to these and other questions made the mathematical ideas both clearer and more interesting. I altered the problem to target those questions (see Figure 2) and have since used the updated version in many contexts from first-year-algebra classes to professional development workshops. This problem, although accessible to most students, provides fertile ground for reasoning and making connections among multiple representations. It also provides a context for the behavior of rational functions. I encourage you to solve the problem before reading on, keeping in mind this tenet from Principles to Actions: Ensuring Mathematical Success for All: "What is critical is that a task provide students with the opportunity to engage actively in reasoning, sense making, and problem solving so that they develop a deep understanding of mathematics" (NCTM, 2014, p. 20).

The Original Problem
This rectangle contains 48 square units, half of which are gray and half of which are white. There is only one other rectangle that can be lined in this same fashion using an equal number of two different colors. Can you find it?
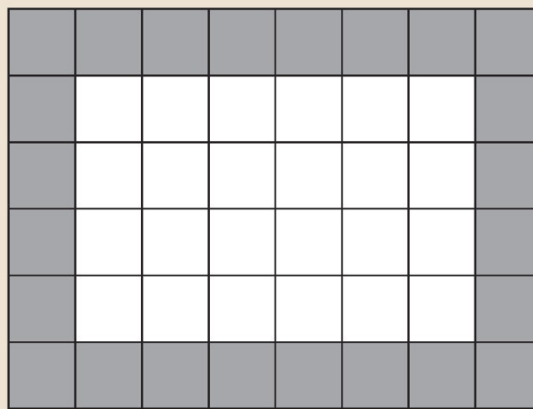
Figure 1: Niederman's statement of the original problem (2009, p. 143) provides more information than the restatement in Figure 2.

Learning while solving a problem is often like entering a dark room with a single small light. The objects are in the shadows, difficult to make out, but as your eyes adjust and more lights go on, the objects and how they sit in relation to one another become sharper. Learning through problem solving occurs from grappling with some idea that we do not completely understand. In search of reasons and connections, we work to clarify, and the objects, ideas, and their relation to one another become clearer. As teachers, we can choose problems that allow students to bring some knowledge to apply to the situation and that challenge them to grapple with new ideas. The remaining sections describe the challenges

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[5] Reprinted with permission from "Illuminating Rectangle Border Challenges," *Mathematics Teacher*, 112(2), copyright 2018 by the National Council of the Teachers of Mathematics (NCTM). All rights reserved.

Rectangle Border (Altered Version)
This rectangle consists of 48 square units in a 6×8 array. If all the squares on the border are shaded, half of the 48 squares are shaded and half of them are unshaded. Are there any other rectangles with natural number side lengths with this property? If so, describe it (them) and show how you found it (them).
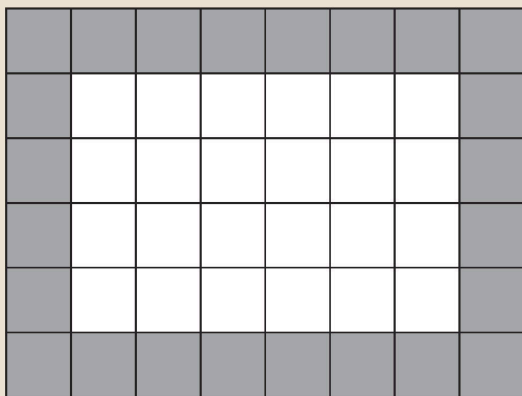
Figure 2: This more open-ended version of the problem was presented to students.

that students have encountered while solving the Rectangle Border problem and how those challenges helped them make sense of ideas that were new to them.

Challenge 1
For most students, guess and check seems like a good way to start the solution process; it gives them a feel for the problem, and guessability makes the problem accessible and motivates them to keep trying to solve it. Many students make guesses with no seeming strategy or reflection, whereas others choose dimensions of a rectangle that are proportional to the given rectangle, usually guessing that a 3×4 or a 12×16 unit rectangle will have the property. Why don't either of those rectangles work? Further, why do students expect a solution to be a rectangle whose dimensions are proportional to the given rectangle? It may be because the problem can be restated in the language of a proportion: the ratio of the number of units in the border to the total number of square units in the rectangle is $1/2$. Nonetheless, this approach leads to a dead end.

Pushed to guess and check wisely and reminded that they will need to justify how they know they have found *all* solutions, students slow down and look for patterns. Some sketch the rectangles on grid paper, which keeps their guesses connected to the context and allows for new insights. From their sketches, a couple of students notice that no rectangle with a width or length of 4 will work. This observation is a nice shift to making conjectures and generalizing. When one dimension is 4, the rectangle has the same number of units in two borders as the number of units in the interior and the other two borders combined (see Figure 3). Shading in the other two borders tips the number of units in the border over one-half. The students conclude that any rectangle with a width or length of 4 or *fewer* will have more units in the border than in the interior.

We pause and imagine several cases: fix one dimension at 4 while the other dimension gets really large. The number of units in the border gets much closer to being half of the total units, but can never be half. This gives us some sense of the relationship between the side lengths and foreshadows, for students thinking graphically, the existence of an asymptote. Imagining inspires further scenarios: What happens if the rectangle is as small as possible? Very
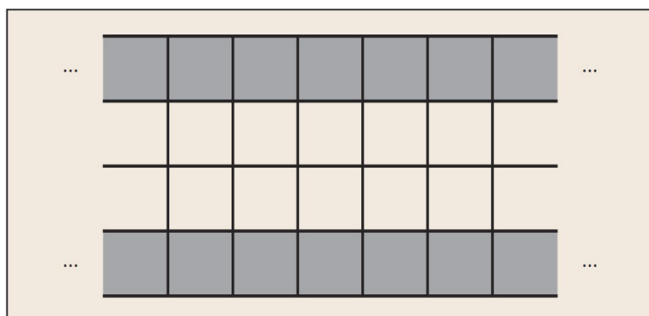

Figure 3: Rectangles with 4-unit lengths will always have more than one-half of their units shaded.

large? "Skinny" versus "fat?" Encouraging students to experiment, visualize, and articulate their ideas about these possibilities can result in engaging discussions.

Imagine a rectangle that is 100 units × 100 units. Clearly, the interior has far more units than the border. Alternatively, focus on small changes to reveal patterns: If we imagine a rectangle with a shaded border, then add 1 unit to the length; the border will grow by 2 units, but the interior will grow by 2 fewer than the width. Similar growth takes place when adding 1 to the width. These shifts in perspective illuminate the structure of the relationship between the border and interior of a rectangle. They also allow students to notice that the interior grows much faster than the border—quadratically versus linearly—and understand why the solutions cannot be proportional to the original rectangle.

While still guessing-and-checking wisely, some students create conjectures that they can justify by reasoning with the numbers. They reason that one dimension must be even because if both dimensions are odd, the total number of square units in the rectangle will be odd, and for half to be shaded and half unshaded, there must be an even number of units. Some take it one more step: The number of units in any border is even, so since half of the square units in the rectangle is an even number, the total number of square units in the rectangle must be divisible by 4. This refines the guessing space.

After a time, either through serendipity or the insights above, a group usually finds that a 5 × 12 rectangle works. But since this new solution is not proportional to the 6 × 8 rectangle and there are no obvious connections between the two rectangles, they have only one other solution—no "Aha! This makes sense!" or insights into how to find more solutions or justify how they know they found them all. Guessing, they learn, does not always satisfy their appetite for sense making. Still, listing students' conjectures and observations from this portion of the investigation and revisiting them later can help them better understand the solution and how they can improve later explorations. The real power of this problem comes from the challenges that arise as the class shifts from guessing and checking to algebra and then to functions.

Challenge 2

The second challenge arises from applying algebra. A few students will define variables and write equations before someone in the class has found the 5 × 12 solution. But some students are misled by the term "border" and use the formula for perimeter. To find a correct formula, they must scrutinize the context and create an expression for the border.

Dimensions	Area	outer	inner		
(6×8)	48	24 4×6	24 4×6	6+6+6+6	8-2×6-2
3×4	12	10	2	3+3+2+2	3-2×4-2
12×16	192	52	140	12·2+14·2	12-2×16-2
7×9	63	28		7·2+7·2	7-2×9-2
8×10	80	32		8·2+8·2	8-2×10-2
10×12	120	40	80	2l+2(w-2)	(l-2)(w-2)

Figure 4: One group adds two columns to examine their processes for calculating the number of tiles in the inner rectangle and in the border.

When they are guessing and checking, encourage them to use their tables as mathematical tools for thinking. By committing the expressions that they used to calculate their guesses to paper, they can examine the structures of those expressions and then write out formulas. Figure 4 shows how one group added the two far-right columns after such encouragement and found the general expressions.

Table 1 shows the correct equations written most often by students. In each case, L is the number of units in the length of the rectangle and W is the number of units in its width.

Table 1 Common Correct Equations Produced by Students	
Relationship in Words	**Algebra**
1. Half the total number of the square units is equal to the number of units in the border.	$\left(\dfrac{1}{2}\right) LW = 2L + 2W - 4$
2. The ratio of the number of units in the border to the total number of square units is 1/2.	$\dfrac{2L + 2W - 4}{LW} = \dfrac{1}{2}$
3. Half the total number of square units is equal to the number of units in the interior.	$(L - 2)(W - 2) = \dfrac{LW}{2}$
4. The number of square units in the interior is equal to the number of units in the border.	$(L - 2)(W - 2) = 2L + 2W - 4$
5. The ratio of the number of units in the border to the number of square units in the interior is 1.	$\dfrac{2L + 2W - 4}{(L - 2)(W - 2)} = 1$
6. The total number of square units in the rectangle is equal to the sum of the units in the border and the square units in the interior.	$LW = (L - 2)(W - 2) + (2L + 2W - 4)$

Exploring students' ideas here can provide several opportunities for sense making. One opportunity is to examine the ratios in equations 2 and 5 to see that L and W are added in the numerator but multiplied in the denominator. This confirms that the numerator will grow more slowly than the denominator when both L and W are growing. Another opportunity is to analyze the last equation: The number of units in the rectangle is the sum of the units in the interior and the square units in the border. This is certainly true, but not helpful. What is there to learn from discussing this relationship? The two expressions on either side of the equation are equivalent, so the equation is true for any rectangle whose length and width are each at least 2 units. The equation does not use the relationship that the interior and border must be numerically equal. Comparing equations 1 through 5 with equation 6 is a good way to view the different meanings of equivalent equations and equivalent expressions.

Students most commonly write equations 1, 3, and 4. Lured by seeing multiple equations in two unknowns, many groups pair two of the equations to set up a system and solve for L and W. But what happens? They find $0 = 0$ and conclude "infinitely many solutions." Of course, there are not infinitely many rectangles with this property because of the restrictions on L and W. What they have shown is that the first five equations are equivalent; they all express the same relationship with no new information. The use of algebra does not seem to have gotten them any closer to a solution. So, the challenge is this: What can they do?

Challenge 3
The third challenge for students is to find a productive way forward once they have an equation in two variables. The equation relates two side lengths, and they know at least one

and maybe two solutions, (6, 8) and (5, 12), although some may realize that (8, 6) and (12, 5) are also solutions. To further explore, it is helpful to shift from creating an *equation* that relates the number of border units to the number of interior units to thinking about the relationship as a *function*. What is the length in terms of the width of any rectangle that has this property? Thinking about the relationship as a function allows us to consider inputs and outputs: What natural number inputs result in natural number outputs? It encourages us to use a graph and table and to look at the structure of the relationship between the two dimensions of rectangles with this property.

Using any of the first five equations in table 1, we get the same function, one side length as a function of the other. Which dimension is length and which dimension is width does not matter because solving for L or W results in the same function and shows that the function is its own inverse.

When L is described in terms of W, students usually get an expression equivalent to the right side of

$$L = \frac{4W - 8}{W - 4}.$$

They can graph this function using electronic graphing technology. Alternatively, those who have studied transformations of functions can graph it quickly and describe graphical features from its symbolic form by writing it as transformations on the parent function $y = 1/x$, that is, as

$$L = \frac{8}{w - 4}.$$

To get it in this form, rather than use polynomial long division, which many students may not have encountered, we can use reasoning and structure. We want a factor of $W - 4$ in the numerator, so that it can cancel the denominator, so we rewrite the numerator as $4(W - 4) + 8$ and write the fraction as two terms:

$$L = \frac{4(W - 4)}{w - 4} + \frac{8}{W - 4}$$
$$= \frac{8}{W - 4} + 4$$

Now, the horizontal and vertical asymptotes of the graph are more obvious to students who have studied transformations of functions, and they can see how the graph describes what they discovered when reasoning numerically and sketching the rectangles.

Either expression defining the function can be analyzed to find all solutions numerically. Students may either examine the tables on their graphing devices or may create a table with values of W that result in $8/(W - 4)$ being a natural number. They can then reason that the only natural number values of W greater than 4 that result in $8/(W - 4)$ also being a natural

number are when $W - 4$ is a divisor of 8. The values of W are 5, 6, 8, and 12, which give outputs for L of 12, 8, 6, and 5, respectively.

This is a good time for the teacher to pause and direct students' attention to their early exploration of the problem to make connections across the solution methods. As they noticed earlier, only values for L and W greater than 4 make sense in this situation because if either the length or the width were less than 4, the border would contain more units than the interior. Here, the asymptotes of $L = 4$ and $W = 4$ are as expected from the earlier investigation. Help students grasp a real meaning for them: Neither side length can be 4, but if one side length is very large, the other side length can get very close to 4.

A group presents the graph with one solution sketched on the coordinate plane (see Figure 5) and the illustration makes its own case for why there are no more solutions. Figure 6 follows soon after, showing that for each rectangle that is a solution, two of its sides coincide with the axes, one vertex is at the origin, and the opposite vertex is on the graph of the function. The
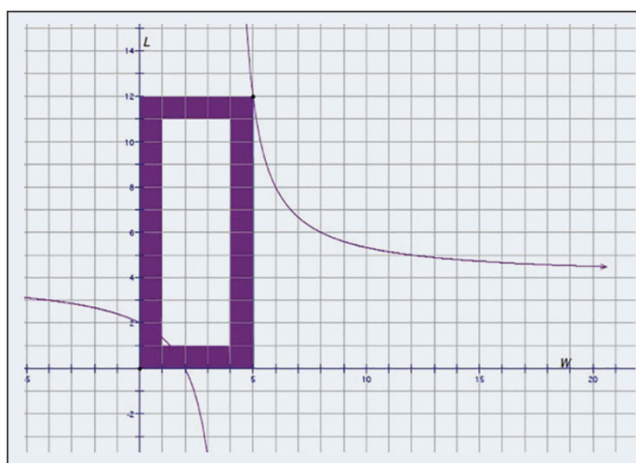

Figure 5: A 5 × 12 rectangle solution, its border shaded, is superimposed on a graph of the function.
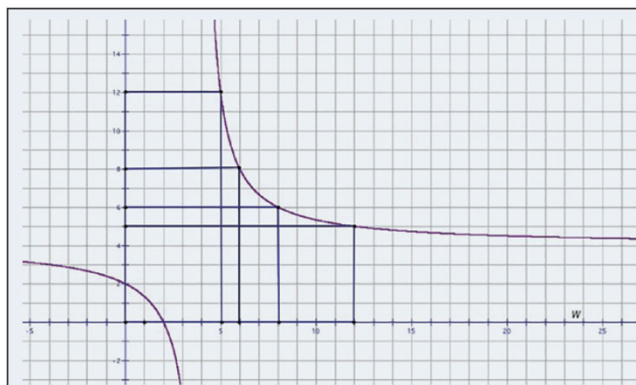

Figure 6: The graph of the function is accompanied by an illustration of all possible rectangles meeting the criteria that L and W must be natural numbers.

function input and output values give the dimensions of the solutions.

To help more students make sense of the graph, continue questioning. First, every point in the first quadrant has meaning for creating rectangles, but only those that are both natural numbers and on the graph where $W > 4$ and $L > 4$ satisfy the constraints of the problem. Ask students to describe what the other points on the graph could mean in the context of the problem. The points on the upper branch of the graph show all rectangles with positive real number side lengths whose interior and 1-unit-wide border have the same area. Note that shifting the discussion of the square units as *objects* to square units as *area* is a conceptual leap for some students. Ask students to use the graph to describe all rectangles whose shaded border is fewer square units than its interior. This further challenges them to make sense of the graph. Any rectangle with two sides on the axes, one vertex at the origin, and whose upper-right vertex lies above the graph of the function will have a smaller border than the interior (see Figure 7a).

Similarly, ask students to describe all rectangles whose shaded border contains more square units than its interior. Rectangles placed like those above but whose upper-right vertex lies
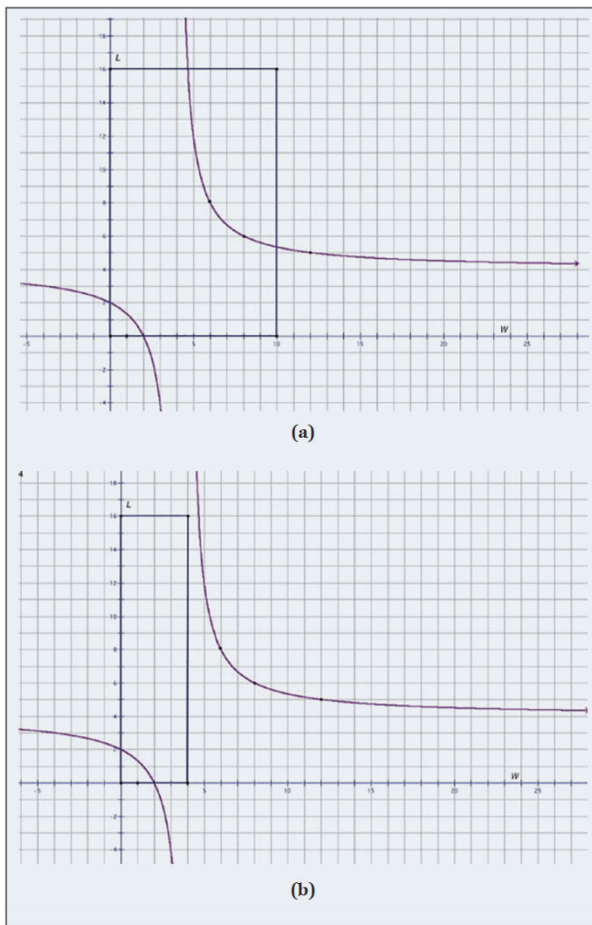
Figure 7: Rectangles whose border area is smaller or larger than its interior area are respectively sketched with the graph in (a) and (b).

below the graph of the first quadrant branch of the hyperbola will have a shaded border that is larger than its interior (see Figure 7b).

Conclusion

The Rectangle Border problem yields many challenges through an accessible context. These challenges generate occasions for students to understand more fully how the mathematics models the problem situation. Students make conjectures that are later confirmed through analyzing structure and multiple representations. Finding an answer with algebra can be unsatisfying when used only as a blunt instrument. But in this problem, when problem solving is slowed down, connections can be made among conjectures, the problem situation, and the graph, and a function can illuminate the relationships. Changing the form of the equation and thinking of it as a function gives clear connections to the situation.

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National Council of Teachers of Mathematics. (2014). *Principles to actions: Ensuring mathematical success for all*. Reston, VA: NCTM.

Niederman, D. (2009). *Number freak: From 1 to 200, the hidden language of numbers revealed*. New York: Penguin Group.

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Escher Art with Hexagons

Timothy Sibbald and Miranda Wheatstone

M. C. Escher was an extraordinary artist best known for his impossible constructions and tessellations.[1] His famous "transformation" prints featured tessellations where a shape was repeated, or tiled, like a checkerboard, but with subtle alterations that changed the subject that the shape was representing. The prints inspired a popular school activity referred to as "Escher art" that involves using a square as a template for tessellation. Patterns are made by cutting out two sides and affixing them to the other two sides so that a regular tessellation is achieved. The patterns in tessellating shapes (SS8.4, Ministry of Education) provide a unique and engaging opportunity for student-led investigative play in the middle grades. As we will see, these investigations can grow significantly in rigor and become suited for advanced students in the upper grades. This article explores what happens when tiling templates are built from hexagons.

Creating a Template

A hexagon can be labeled with any side chosen initially. For our purposes, we will label the topmost side A. Other sides are labeled in a clockwise manner, creating B, C, D, E, and F, as shown in Figure 1. In the Escher art scenario, a pattern is cut in a continuous line from one vertex of a side to the adjoining vertex. The piece that is cut out is then affixed to another side of the hexagon. Figure 2 demonstrates a possible scenario in which the dashed line corresponding to side A shows a cut and the shaded region becomes a separate piece that has been translated and affixed to side D.
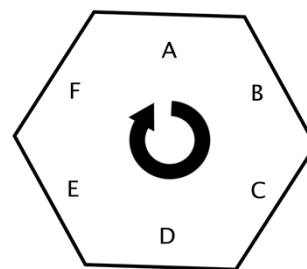

Figure 1: Hexagon labeling

This placement creates a link between sides A and D that allows duplicates of the template to tessellate, with Side A of one piece fitting together with side D of a neighboring piece. This is shown in Figure 3. It is important to notice that the translation of the cut piece allows a new template to be adjoined to the original and that this adjoined template will also have clockwise lettering. It should also be noted that there are currently unlabeled hexagons in the figure, because the template has not determined what happens with sides B, C, E, or F.
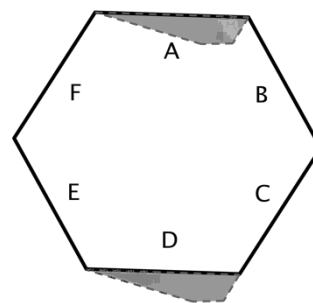

Figure 2: Hexagon template showing cut side and affixed side

An alternative template connecting A and D can be made by affixing the piece to side D in a different manner, by flipping A prior to affixing it to side D. The impact of this action is that the template adjoined to side D is flipped upside down—that is, it will have a reversed orientation, with the sides labeled in a counter clockwise manner. This is shown in Figure 4 and leads to alternating orientations as more pieces are adjoined.

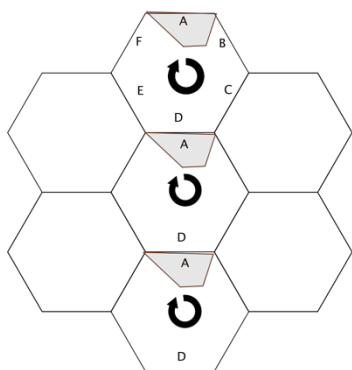[1] You can explore M. C. Escher's work on the official Escher website, www.mcescher.com.

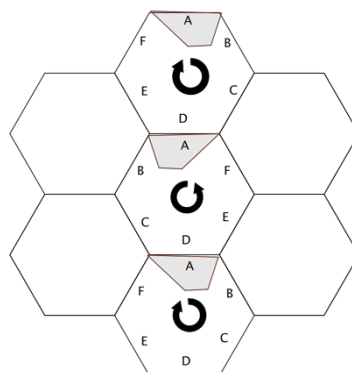*Figure 3: Tessellating the template*

Figure 4: Tessellation when cut is translated and reflected before affixing

this produces a regular lattice pattern with regularly spaced holes.

Using the template from Figure 7, except flipping side A before affixing it to side F, results in a variety of intriguing and exciting tessellations. It is important to note that variety can happen because tessellation involves choices, and some choices constrain the tessellation process (e.g., the holes that cannot be filled in Figure 8). This is an excellent investigative activity for students, as exercising choice can lead to different end results.

An activity sheet is provided on the next page as a possibility for structuring the activity in your classroom. We suggest printing the first page on cardstock, as well as having a large number of hexagon templates available to cut out for further investigations. Having students explore, whether in the suggested direction or more generally, is sure to make for an interesting gallery walk where students can see a variety of possible patterns and generate further questions.

So far, the examples illustrated have involved cutting only one side. Typically, however, half of the sides of the hexagon are cut out and the pieces cut out are affixed to the remaining sides. When these templates are tiled, there are innumerable possibilities owing to the fact that the remaining straight edges can tessellate with one another without restriction. For the earlier example, two additional sides are cut, each from vertex to vertex, and the pieces affixed to the two remaining straight sides. In Figure 5, we have cut out two adjacent sides and rotated them to affix them to the next two adjacent straight sides. Sides A and B were chosen for this example and they were affixed to sides F and C respectively. This template tessellates regularly, filling the plane completely as seen in Figure 6.

Further exploration can be conducted with this template by affixing the cut sides to different sides, such as E and D, or flipping them upside down before affixing them. An investigation in the classroom may encourage students to experiment with various cut-outs in various orientations affixed to different sides and to notice the patterns created. For example, in Figure 7, we have cut out two non-adjacent sides, in this case A and C, and rotated and affixed them to sides F and D, respectively. The tiling of this template creates a hole (grey hexagons) with the inside edges all being smooth, as seen in Figure 8. If continued,
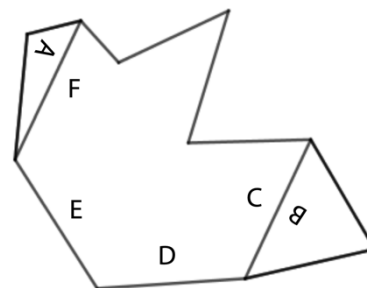

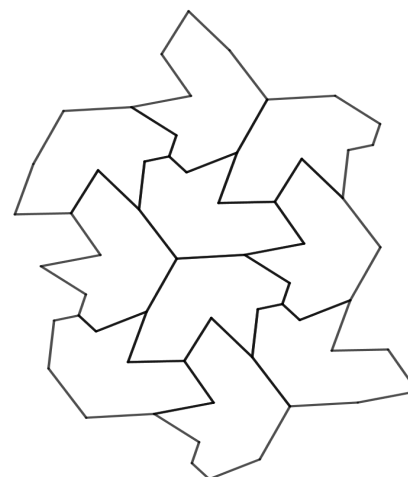*Figure 5: Cutting two adjacent sides*

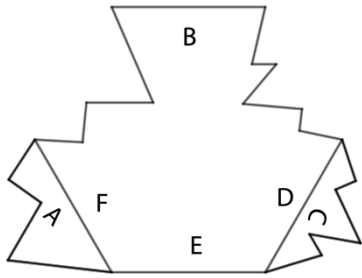*Figure 6: Tiling obtained from the template in Figure 5*

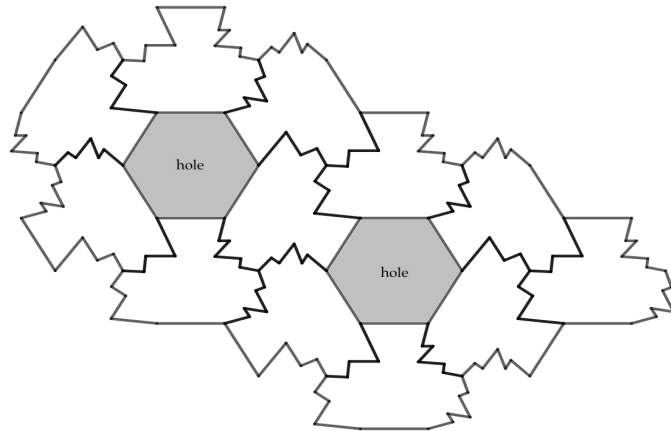*Figure 7: Cutting two non-adjacent sides*

Figure 8: Tiling creating a regular lattice with holes

In the spirit of M.C. Escher, once students have created their tessellation, they can use their artistic spirit to create masterful artworks. They can be as simple or as elaborate as their imaginations allow. Two examples showing Miranda's artistic spirit are provided below.
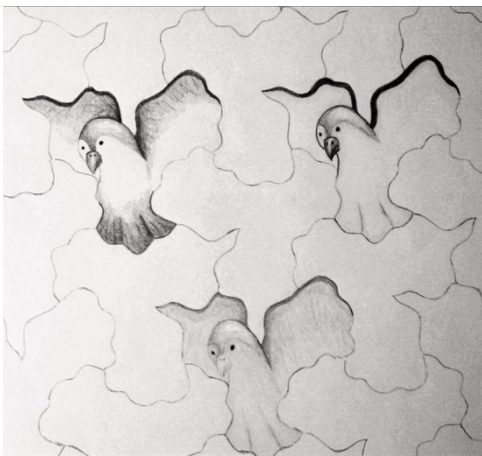
Hexagon Art

In this activity, we are going to create some art using hexagons.
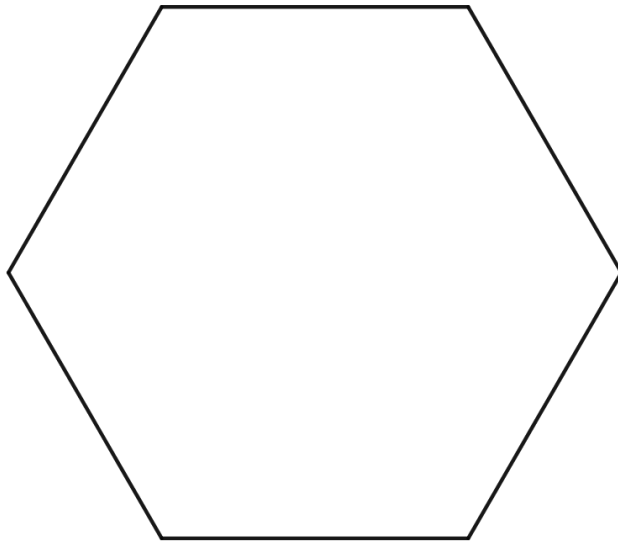
1. Cut out the hexagon.
2. Choose three sides.
3. On each of your chosen sides, trace a shape from corner to corner. Each shape must include the entire side and the three shapes cannot overlap.
4. Cut out the three shapes.
5. Tape each cut-out to any of the other uncut sides. You can tape the cut-outs right-side-up or upside-down.

Here's an example:
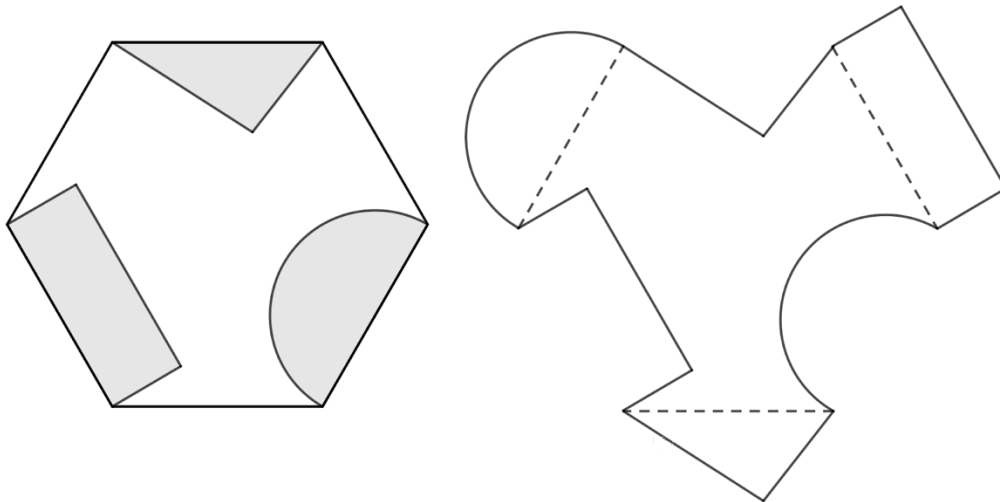
On a separate sheet, trace your new template. Move the template to "fit" with the tracing so that there are no gaps between the template and the tracing. Trace your template again. Repeat this process until your page is filled. If there are spaces where the template does not fit, shade them.
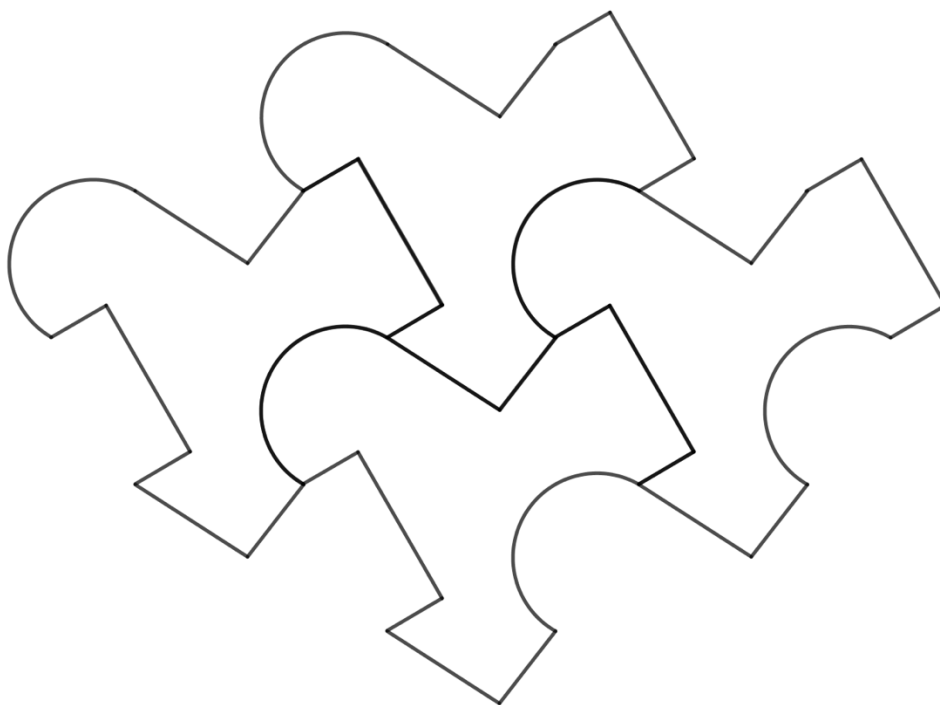
Can you build a template that fills the entire space without leaving any gaps?

Can you build a template that *cannot* fill the entire space without leaving any gaps?

Two further questions

As students investigate different ways of creating templates, they may begin to ask how many unique possibilities exist. This is the question that we have posed in our own research. More precisely, we asked: How many possible (unique) ways are there to affix the sides, and how do the resulting templates behave? Since there are only three regular polygons that tessellate regularly to fill the plane, we began with the square and triangle (Sibbald & Wheatstone, 2016); here, we extend the work to hexagons.

How many templates are possible?

When we began our research into possible hexagon tessellations, we first looked for an orderly way to work through all possible scenarios. In the case of hexagons, it is a good investigative problem for students in higher grades to simply examine the number of ways three sides can be cut out and affixed to three uncut sides. Since the template can be freely rotated, one can always begin with A at the top and assume it has been cut out. In the examples above, we considered two ways that it can then be affixed to side D. For simplicity of notation, we label the simple translation as AD and the reflected version as A-D, where the negative sign indicates that the adjoining template must be turned over, or face down, to be connected.

Considering where A can be adjoined leads to the possibilities AB, A-B, AC, A-C, AD, A-D. Other moves are equivalent to these. For example, AF is a reflection of AB and not genuinely unique. The challenge then becomes to consider, for each of these cases, how many ways the remaining four sides can be used to create novel templates. Consider, for example, that when AD or A-D is used, side B can be cut and affixed to C, -C, E, -E, F, or –F. When this is done, two sides will remain untouched and one can be cut and affixed to the other with a translation or with a translation and reflection (i.e., a negative sign). However, the number of possible combinations is beginning to get confusing and there is a loss of clarity regarding whether cases are inadvertently being duplicated.

An alternative approach is to consider how many ways the sides can be associated geometrically. There are three initial associations: AB, AC, and AD. After a second association, a third association is implied. This is illustrated in Figure 9, where the first association is shown with a line segment that has points at each end and the possible secondary associations are shown with line segments that do not have points at the ends. The rightmost template has three lines emanating from B, which indicate that B could be associated with any of the remaining three sides without resulting in duplicate templates.
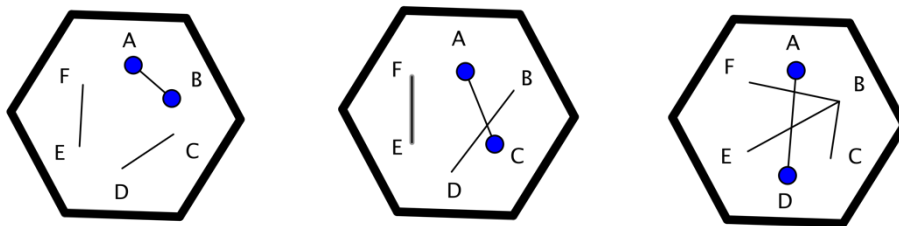

Figure 9: Possible association of sides for templates

With the three scenarios for associations, the question become: How many ways can each of the three scenarios be constructed? This will depend on how many sides are simply translated and how many are reflected as well. This can be addressed by considering each scenario with 0, 1, 2, or 3 sides reflected. In the first scenario (where the initial association is AB), there is one way to make the template with no reflections (AB, CD, EF), which is shown on the left in Figure 10. With one side reflected there is also only one way (A-B, CD, EF), which is shown on the right in Figure 10 (stippling indicates reflection), because rotating the template makes the other possibilities (AB, C-D, EF and AB, CD, E-F) equivalent.

With two sides reflected, no matter which two are selected they are necessarily side by side so, again, there is only one possible arrangement (A-B, C-D, EF). For three reflected sides there is only one possible arrangement (A-B, C-D, E-F). This type of reasoning was used for each
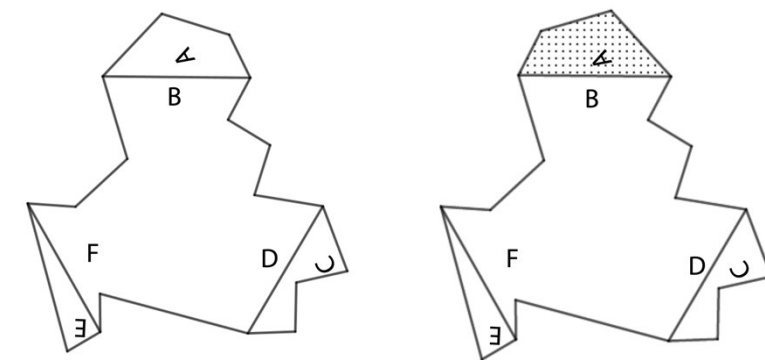
Figure 10: Sample of Scenario 1 with no sides flipped and with one side flipped

of the three scenarios from Figure 9 and is summarized in Table 1. Note that the third scenario was treated as three separate cases and the values in the table sum the number of ways for each number of reflected sides. The result is a grand total of 26 templates.
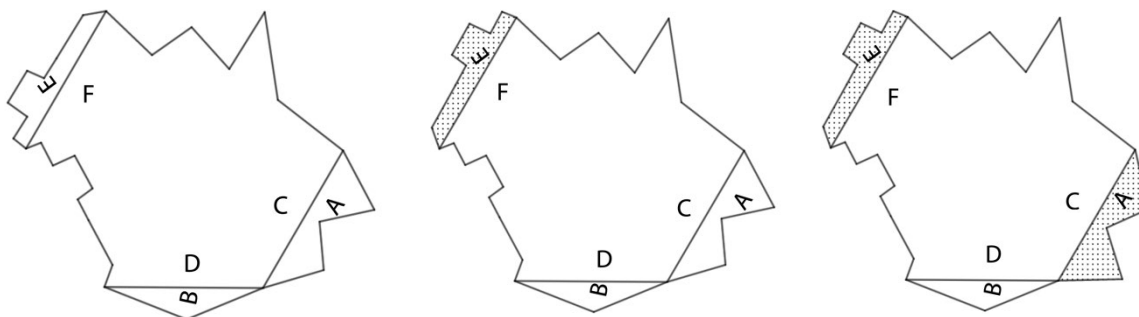
Figure 11: Sample template of Scenario 2 with no sides flipped, one side flipped, and two sides flipped

Table 1
Number of ways to build hexagonal templates.

Scenario	Number of reflected sides				Total number of templates
	0	1	2	3	
1	1	1	1	1	4
2	1	2	2	1	6
3	3	5	5	3	16

In the classroom, the problem of counting the templates is very much an exercise in organizing one's thinking in order to ensure that all scenarios and cases have been considered. It is a wonderful example of how communication can be the crux of a problem-solving task.

How do the templates behave?
With the scenarios determined, our research focused on characterizing what can happen in the different cases. Some templates, such as AD, BE, CF, will tile and fill the entire plane. However, we knew from early explorations mentioned in Sibbald and Wheatstone (2016) that holes and other unusual behaviors were plausible. In particular, with square templates we demonstrated that holes could arise and that it is even possible to make a pattern that is aperiodic (i.e., does not repeat).

The categories that arose from this investigation were: Regular Tessellation (Reg), Hole or Ring (Hole/Ring), Fingers (Fing), and Line (Line). A "regular" tessellation refers to a tiling that is continuous and has no holes (See Figure 6). A "hole" refers to a situation where the tessellation completes a circuit around a single hexagon, but the template cannot fill the middle (see Figure 8). A "ring" is similar to a hole, except that it is a circuit around more than one contiguous hexagon (See Figure 12). "Fingers" refer to having multiple linear tessellations that can continue indefinitely. In some cases, additional offshoots that can continue indefinitely can be added, and we refer to this situation as "hairy fingers"; see Figure 13. Finally, a line is a linear tessellation that does not allow any tessellation away from the line into the second dimension. With these characteristics, we examined the 26 possible cases. The results are shown in Table 2.
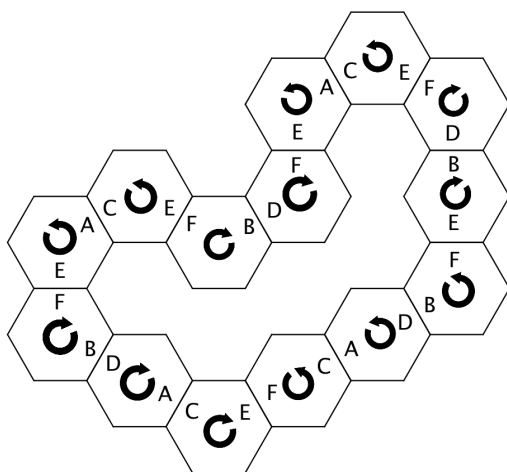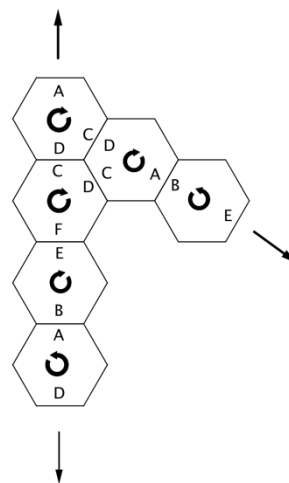

Figure 12: A ring tessellation using AB, CD, E-F

Figure 13: Example of a finger with an offshoot – a "hairy finger"

Table 2
Characteristics behavior arising from the 26 templates.

Template	Reg	Hole/Ring	Fing	Line	Comments
Scenario 1					
AB, CD, EF	X				
A-B, CD, EF		X	X		Hairy fingers (allows offshoot fingers). See Figure 13.
A-B, C-D, EF		X	X		Finger allows offshoots on one side only.
A-B, C-D, E-F		X	X	X	
Scenario 2					
AC, BD, EF		X			
A-C, BD, EF		X	X		Hairy fingers (allows offshoot fingers).
AC, BD, E-F		X			See Fig. 12. Possible spiral? See Fig. 14.
A-C, B-D, EF		X	X		Hairy fingers (allows offshoot fingers).
A-C, BD, E-F		X		X	Two different ways to make a straight line that cannot be added to. Examples have been made with hole/ring sizes 1, 2, and 3 hexagons.
A-C, B-D, E-F		X		X	
Scenario 3					
AD, BC, EF		X	X	X	
A-D, BC, EF		X		X	
AD, B-C, EF		X	X	X	
A-D, B-C, EF		X	X	X	
AD, B-C, E-F	X				
A-D, B-C, E-F		X	X	X	
AD, BE, CF	X				
A-D, BE, CF		X	X	X	Two different ways to make a straight line that cannot be added to. Holes can be made with sizes 2, 4, 6, … hexagons.
A-D, B-E, CF		X	X	X	
A-D, B-E, C-F		X	X	X	
AD, BF, CE		X	X	X	
A-D, BF, CE		X	X	X	Two different ways to make a straight line that cannot be attached to.
AD, B-F, CE		X	X	X	Two different ways to make a straight line that cannot be attached to.
A-D, B-F, CE		X	X	X	
AD, B-F, C-E	X				
A-D, B-F, C-E		X	X	X	

The findings show a variety of behaviors and an activity in which every student in a class could have a unique template to study. The teacher could also assign templates to provide tiered problem solving that addresses individual needs of students. Moreover, there is also room for enrichment! We have not exhausted what can happen when these shapes are tessellated. Consider that the AC, BD, E-F tessellation was used to develop the possible spiral shown in Figure 14. We say "possible" because there is the important detail of showing that as the pattern continues it will never be feasible to connect the outer arm to the next arm toward the center. This is a first-rate example of the challenge of proving a result, rather than simply accepting the given evidence. Perhaps more importantly is the opportunity for a teacher to teach students the very important lesson of sharing that "We don't know."
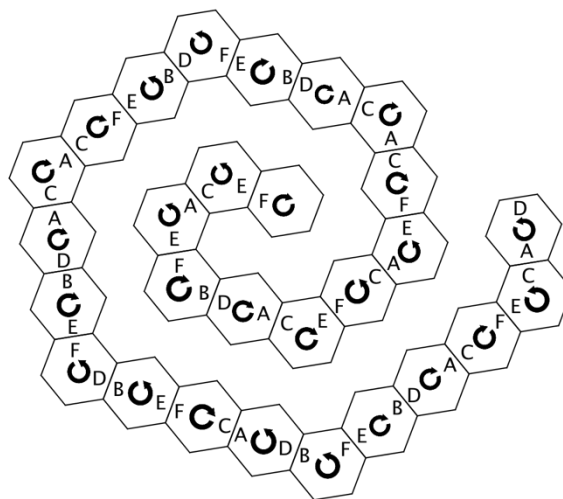

Figure 14: A hexagonal tessellation spiral

Tessellating hexagons provides a significant opportunity for geometric problem solving with embedded patterns. It goes beyond the remarkable properties of tessellating squares and has opportunities for students to explore and report on. There are many opportunities to develop a variety of problem-solving skills, such as communication, the use of different strategies, creating and working with different representations, and more. It is the sort of activity that will definitely enrich a classroom math community.

References

Sibbald, T. M., & Wheatstone, M. (2016). Advancing Escher art through generalization. *Ontario Association for Mathematics Education Gazette, 54*(4), 23–26. Available at http://faculty.nipissingu.ca/timothys/pub/SibbaldWheatstone2016.pdf

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Dr. Timothy Sibbald is an associate professor at the Schulich School of Education at Nipissing University. His interests focus on classroom instructional issues, content development and delivery, and teacher development. He is the editor of the Ontario Association for Mathematics Education Gazette.

Miranda is a graduate from Nipissing University, where her interest in Escher Art was sparked. She is currently working as a teacher for the York Region District School Board.

Intersections

In this column, you'll find information about upcoming math education-related workshops, conferences, and other events. Some events fill up fast, so don't delay signing up!

For more information about a particular event or to register, follow the link provided below the description. If you know about an event that should be on our list, please contact us at thevariable@smts.ca.

Within Saskatchewan

Technology in Mathematics Foundations and Pre-Calculus
March 7, 2019
Regina, SK
Presented by the Saskatchewan Professional Development Unit

Technology is a tool that allows students to understand senior mathematics in a deeper way. This workshop is designed to have math foundations and pre-calculus teachers experience a variety of technology tools that allow students to represent and visualize mathematics concepts. Tools highlighted are useful for students to explore, learn, communicate, collaborate and practice, in order to enhance their understanding of mathematics in secondary mathematics.
More information at https://www.stf.sk.ca/professional-resources/events-calendar/technology-mathematics-foundations-and-pre-calculus

Accreditation Renewal/Second Seminar
March 7, April 12, 2019
Saskatoon, SK
Presented by the Saskatchewan Professional Development Unit

Accreditation seminars are offered to enable qualified teachers to become accredited. Accreditation is the process by which qualified teachers are granted the responsibility of determining the final mark or standing of the students in a specified Grade 12 (level 30) subject or subjects.

More information at https://www.stf.sk.ca/professional-resources/events-calendar/accreditation-renewalsecond-seminar-0

Making Math Class Work
April 11, 2019
Tisdale, SK
Presented by the Saskatchewan Professional Development Unit

Math classrooms across Saskatchewan are increasingly complex and diverse. Meeting everyone's needs can be daunting, even with all of the instructional strategies and structures available to teachers. Number Talks, Guided Math, Rich Tasks, Problem Based Learning, Open Questions, High Yield Routines are just some of the strategies available to teachers, but where to start? Come work collaboratively to problem solve how to make math class work for you and your students.

More information at https://www.stf.sk.ca/professional-resources/events-calendar/making-math-class-work-0

Early Learning With Block Play – Numeracy, Science, Literacy and So Much More!
April 12, 2019
Moose Jaw, SK
Presented by the Saskatchewan Professional Development Unit

This is a one-day workshop for early learning educators from prekindergarten, kindergarten and Grade 1 to work collaboratively to discover and deepen their understandings around the many foundational skills that children develop during block play. Through concrete, hands-on activities, participants will experience and examine the many connections between block play and curricular outcomes, and the current research on the topic.

More information at https://www.stf.sk.ca/professional-resources/events-calendar/early-learning-block-play-numeracy-science-literacy-more

Beyond Saskatchewan

NCTM Annual Meeting and Exposition
April 3-6, 2018
San Diego, CA
Presented by the National Council of Teachers of Mathematics

Join thousands of your mathematics education peers at the premier math education event of the year! Network and exchange ideas, engage with innovation in the field, and discover new learning practices that will drive student success. Classroom teachers, administrators, math coaches, supervisors, college professors, and preservice teachers can all benefit from the sessions and learning at this event.

Head to http://www.nctm.org/Conferences-and-Professional-Development/Annual-Meeting-and-Exposition/

46th Annual OAME Conference: All In
May 16-18, 2018
Ottawa, ON
Presented by the Ontario Association for Mathematics Education Conference

Join hundreds of your mathematics education peers in Ottawa, Ontario for the 46th Annual OAME Conference. This year's featured speakers include Marian Small, Eli Luberoff, Jules Bonin-Ducharme, Tracy Zager, Nat Banting, and many more!

For more information, head to https://oame2019.ca

Online Workshops

Education Week Math Webinars
Once a month, Education Weekly has a webinar focusing on math. They also host their previous webinars on this site. Previous webinars include Formative Assessment, Dynamic vs. Static Assessment, Productive Struggling, and Differentiation.

Past webinars: www.edweek.org/ew/webinars/math-webinars.html
Upcoming webinars: www.edweek.org/ew/marketplace/webinars/webinars.html

Tangents
Extracurricular Opportunities for K-12 Students

*T*his column highlights local and national extracurricular opportunities for K-12 students interested in mathematics, including collaborative, individual, online, and in-person challenges, contests, and camps. For dates, registration procedures and applications, and other information about the contests listed, please head to the contest websites, included below the descriptions. If we have missed an event that should be on our list, please contact us at *thevariable@smts.ca*.

If you are looking for contests available in other provinces, head to the Canadian Mathematical Society website (cms.math.ca/Competitions/othercanadian). The CMS also maintains a list of resources for students who are looking to build their problem-solving skills and succeed in competitions: see cms.math.ca/Competitions/problemsolving.

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Canadian Math Kangaroo Contest
March 24, 2019

The purpose of this competition is to introduce youngsters from Grade 1 to Grade 12 to math challenges in a fun and enjoyable way, thus inspiring their further interest and advancement in mathematics. The competition is held yearly in more than 40 Canadian cities. Students may choose to participate in English or in French.

More information at kangaroo.math.ca/index.php?lang=en

Caribou Mathematics Competition
Held six times throughout the school year

The Caribou Mathematics Competition is a worldwide online contest that is held six times throughout the school year. Each of these days, five contests are offered, one for each of the grade levels 3/4, 5/6, 7/8, 9/10 and 11/12 and each one in English, French and Persian. Available in English, French, and Persian.

More information at cariboutests.com

Euclid Mathematics Contest
Written in April
Presented by the Centre for Education in Mathematics and Computing (University of Waterloo)

The Euclid Mathematics Contest is an opportunity for students in their final year of secondary school and motivated students in lower grades to have fun and to develop their mathematical problem solving ability. Most of the problems are based on the mathematical curriculum up to and including the final year of secondary school. Most of the problems are based on curricula up to and including the final year of secondary school. Some content might require students to extend their knowledge and the best way to familiarize oneself with commonly appearing topics is to practice using past contests.

More information at www.cemc.uwaterloo.ca/contests/euclid.html

Fryer, Galois, and Hypatia Mathematics Contests
Written in April
Presented by the Centre for Education in Mathematics and Computing (University of Waterloo)

The Fryer, Galois and Hypatia Math Contests are an opportunity for students to write a full-solution contest. They are fun way to develop mathematical problem solving skills through a written mathematical activity. For students in Grades 9 (Fryer), 10 (Galois) and 11 (Hypatia). Questions are based on curriculum common to all Canadian provinces. Rather than testing content, most of the contest problems test logical thinking and mathematical problem solving.

More information at www.cemc.uwaterloo.ca/contests/fgh.html

Gauss Mathematics Contests
Written in May
Presented by the Centre for Education in Mathematics and Computing (University of Waterloo)

The Gauss Contests are an opportunity for students to have fun and to develop their mathematical problem solving ability. For all students in Grades 7 and 8 and interested students from lower grades. Questions are based on curriculum common to all Canadian provinces.

More information at www.cemc.uwaterloo.ca/contests/gauss.html

Opti-Math
Written in March
Presented by the Groupe des responsables en mathématique au secondaire

A French-language mathematics challenge for secondary students who would like to exercise and develop their problem-solving skills.

More information at www.optimath.ca/index.html

Pascal, Cayley, and Fermat Contests
Written in February
Presented by the Centre for Education in Mathematics and Computing (University of Waterloo)

The Pascal, Cayley and Fermat Contests are an opportunity for students to have fun and to develop their mathematical problem solving ability. For students in Grades 9 (Fryer), 10 (Galois) and 11 (Hypatia). Early questions require only concepts found in the curriculum

common to all provinces. The last few questions are designed to test ingenuity and insight. Rather than testing content, most of the contest problems test logical thinking and mathematical problem solving.

More information at www.cemc.uwaterloo.ca/contests/pcf.html

The Virtual Mathematical Marathon
Supported by the Canadian National Science and Engineering Research Council

The virtual Mathematical Marathon has been developed by an international team of mathematicians, mathematics educators and computer science specialists with the help of the Canadian National Science and Engineering Research Council and its PromoScience program.

The main activity is a competition allowing students to enjoy solving challenging mathematical problems all year around. Students can join the game at any time and at no cost, simply creating an individual profile with an individual username and a password. Available in French.

More information at www8.umoncton.ca/umcm-mmv/index.php

Problems to Ponder

Sale

Two competing shops have shoes for sale, and both are asking the same price. Both shops have a sale; the first shop drops the price of the shoes by $18, the second drops it by 18%. The following week, the first shop drops the prices of the shoes by a further 21%, while the second shops takes off a further $21. After this second round of reductions, the two shoes are again offering the shoes at the same price.

What was the original price of the shirt, in dollars?

Source: http://mei.org.uk/month-item-17

Exponential Percentages

- If a population grows by 10% of its current size each month, how long will it take to double its size?
- If a population shrinks by 10% of its current size each month, how long will it take to halve its size?
- If a population alternately grows and shrinks by 10% each month, what happens in the long run?

Source: Mason, J., Burton, L, & Stacey, K. (1985). *Thinking mathematically.* Essex, England: Prentice Hall.

Math Ed Matters by MatthewMaddux *is a bimonthly column telling slightly bent, untold, true stories of mathematics teaching and learning.*

Consumeracy: Consumer Numeracy

Egan J Chernoff
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Making the perfect pizza purchase is a scenario you've probably analyzed in math class. I've done it, too. Here's a typical set-up:

A restaurant offers an extra-large pizza that is circular in shape for some price. A competing restaurant offers an extra-large pizza that is rectangular in shape for a different price. Which is the better deal?

The scenario leads to a discussion about whether the dimensions of rectangular pizzas represent the side length or the diagonal and whether the dimensions of circular pizzas refer to the radius, the diameter, or the width of the box. Then, once these questions have been resolved: Cue the calculations! A likely strategy is to compare the differences in price relative to the area for both pizzas in order to determine, theoretically, the best bang for our buck.

There's just one issue with this scenario: it's not realistic.[1] A more likely pizza-purchasing scenario involves deciding whether the possible sides (e.g., bread sticks or wings) or the dipping sauces (e.g., ranch vs. blue cheese) or the 2 L bottle of soda would enhance the pizza dining experience and would therefore be "worth it," despite any changes in the total price. The pizza related accoutrements, thus, render any nuanced difference in area-based-pizza-prices as moot. Sure, there's some merit, I guess, in comparing the differences between the price of rectangular pizza with bacon-wrapped edges versus a circular pizza that has either cheese or hotdogs or both stuffed inside the crust. But let's be honest: such nuanced comparisons just don't happen.

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[1] See Boaler's *What's Math Got To Do With It?* for a popular account of pseudocontext in the math class.

There's a least one more issue with the pizza example: the implication that getting the most fast-food for the cheapest price is the best-case scenario. By and large, we probably get a pass on this because we live in North America, but I'm pretty sure that if you went around trumpeting how your class just calculated that two XL hot-dog-and-cheese-stuffed-crust pizzas, a couple dozen chicken wings, breadsticks, some ranch and blue cheese dressing, and 4 L of Mountain Dew to wash it all down, is the *best deal*… you would get some mixed reactions. I suspect that the school nurse, or your colleague who teaches health sciences, might want to have a few words about the "lessons" you're teaching in math class.

> "Many of the consumeracy examples that I've stumbled upon over the years require no small measure of ignoring common sense."

For this reason, this example, and many of the other *consumeracy* (my portmanteau of *consumer* and *numeracy*) examples that I've stumbled upon over the years have been a little hard for me to swallow, because they require no small measure of ignoring common sense. With this in mind, let's look at a more realistic situation.

My favourite consumeracy example is one that I've used for years, that I still use it to this day, and that I plan to keep using in the future. Here's a version of the problem[2]:

> Spring is here and last year's parkas are on sale! The parkas are 35% off, but, as you know, we pay an 11% sales tax in Saskatchewan. As you head to the counter with your new coat tucked under your arm, your friend asks: "Hey, what should we tell them to calculate first: discount or tax?"

Admittedly, this example is not free of contrived context. Do parkas really go on sale in the spring in Saskatchewan? Yes. Is the provincial sales tax really 11%? Currently, yes. Are end-of-winter discounts really around 35% off? Yes and no; that is, some discounts are worse and some are better. Does Saskatchewan really experience spring? In my opinion, no. So, yes, this problem also requires some suspension of disbelief, but at the very core of the problem is a realistic question. In other words, you can change the context all you want, but the central question—that is, would you prefer the teller to first calculate the tax or the discount—remains.

In what follows, it is assumed that you're either familiar with the problem or have just worked out the solution. In other words: Spoiler alert!

Most of the time, when I pose this problem, people immediately fall into one of two camps. First, there are those who want the tax calculated first and then the discount. When pressed for an explanation, the *tax-then-discount* proponents, especially those who haven't delved into any calculations at this point, explain that if you add the tax first, you will be getting a larger discount on your item (e.g., 35% off $200 is more than 35% off $100). *Discount-then-tax* proponents, typically the majority, are quick to rebuke the others. Explanations usually fall in line with some version of, "Look, you want them to take off the discount first because then, in the end, you'll be paying less tax." Again, not having delved into calculations, explanations focus on how paying tax on less money equates to paying less tax (e.g., 11% tax on $100 is more than 11% tax on $50). There are also some less likely responses.

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[2] I first came across this example, entitled Warehouse, in *Thinking Mathematically* by Mason with Burton and Stacey.

Some of these responses still fall into either the tax-then-discount or discount-then-tax camps, but take the perspective of the shop owner rather than the consumer. From this perspective, on the one hand, *tax-then-discounters* will explain how it is important to capture the tax on the full amount of money that the item costs before one applies any discount. On the other hand, *discount-then-taxers* explain that you do not want to discount any of the tax that the customer pays, which means that you should calculate the discount first and then get the full amount of tax on the discounted price.

Having thought about this consumeracy example for some time now, I believe that the manner in which the question is asked has a lot to do with the surprise associated with the correct answer. (After all, when presented with two options, a third option will often be a surprise.) Usually the unexpected, correct answer to the question comes from a person or group of people who have been silent and scratching out a few calculations while the others were busy proclaiming, one way or the other, the manner in which they want to deal with the tax and the discount. With no small degree of self-satisfaction, someone will let the rest of the room know that it doesn't matter whether the tax or the discount is calculated first, because in both cases, you get the same answer. Although you might think that that this would put an end to the conversation, it is often only the beginning.

Whether dealing with this example in a more traditional lecture format or setting up the task so that the students are, in essence, driving the bus, the same notions are covered. First, there are cases and to examples consider. Perhaps the skeptics in the room are still not convinced. Great! An opportunity to consider different examples with different dollar amounts. Barring any calculation errors that might occur, the examples should reveal a pattern amongst the varying examples and cases. While the room may have less skeptics than before, the more general question of "why" still remains.

In what might be called the after-phase of the discussion, the following calculations may be found on the whiteboard at the front of the room:

<div style="text-align:center">

Tax-then-Discount
Cost of item: $100
Tax: 10%
Discount: 20%

Calculate Tax: 10% of $100 is $10
Add Tax to Price: $100 and $10 is $110

Calculate Discount: 20% off $110 is $22
Subtract Discount from Taxed Price:
$110 less $22 is…

$88

Discount-then-Tax
Cost of item: $100
Discount: 20%
Tax: 10%

Calculate Discount: 20% off $100 is $20
Price less Discount: $100 less $20 is $80

Calculate Tax: 10% of $80 is $8
Add Tax to Discounted Price:
$8 and $80 is…

$88

</div>

In an effort to get everybody in the room to the same place, a neat and tidy presentation of two cases, utilizing nice round numbers for the calculations, may convince earlier skeptics. Others, though, particularly those who do not fall prey to fallacy of appealing to authority, may have only renewed their skepticism. After all, the larger question of "why" still remains.

It's at this point that this consumeracy example provides a nice opportunity to demonstrate that calculating and applying taxes and discounts can be done differently than as presented in the example above. It can be shown (for example, by utilizing a rectangle to represent the full cost of the item) that a discount of 20% off an item is the same as paying for 80% of the original price. Similarly, a discount of 40% off is akin to paying 60% of the original price. Therefore, instead of calculating the discount (e.g., 40% of $100 is $40) and then subtracting it from the original price (e.g., $100 less $40 is $60), one could simply determine 60% of the original price (e.g., 60% of $100 is $60). The same goes for taxes. A tax of 10% is akin to paying for 110% of the item: that is, the original price (100%) and the extra (10%). As with the discount, instead of calculating the tax (e.g., 10% of $100 is $10) and then adding it to the price (e.g., $100 and $10 is $110), one could simply determine 110% of the original price (e.g., 110% of $100 which is $110). Following this conversation, the calculations on the whiteboard now look a little different:

Tax-then-Discount	**Discount-then-Tax**
Cost of item: $100	Cost of item: $100
Tax: 10%	Discount: 20%
Discount: 20%	Tax: 10%
$100(1.1)(0.8)=$88	$100(0.8)(1.1)=$88

The new calculations on the board provide an opportunity not only to discuss how the calculations will hold no matter the cost of the item, but also provide a gateway to discussing the commutative property of multiplication. (I would be remiss not to mention that we have, here, a "real-life" example of the commutative property of multiplication.) As I claimed earlier, this consumeracy example, in my opinion, hits on all cylinders. With so many important notions embedded in this problem (e.g., cases, examples, percentages, patterns, generalizations, the commutative property, etc.), it's important to remember that at its core, the problem is about consumer numeracy—*consumeracy*. And the responses, especially after tasking those involved in the lesson to ask friends and family members whether they have a preference regarding tax and discount, cements, at least for me, the importance of consumeracy.

> "In a world of rapidly rising household debt, mortgage rates, bank fees, and student loans, financial literacy is a critical tool for citizens of the world today."

It's important to point out that school mathematics is not devoid of financial literacy education. Mathematics curricula, textbooks, and lessons are filled with problems involving debt, credit, cash back, simple interest, compound interest, banking, budgeting, buying versus leasing vehicles, loans, mortgages, and more; and I believe that in the not-too-distant future, financial literacy will play an even more prominent role in the mathematics classroom than it does today. In a world of rising household debt (especially credit card debt), rising mortgage rates (after years and years of historically low rates), pervasive predatory lending, rampant bank fees, student loans, and planning for one's future, financial literacy is a critical tool for citizens of the world today. Embedded within financial literacy, but not as prominent in school mathematics, is the mathematics of consumerism.

Simply put, consumerism is becoming more mathematically complicated. Take for example the case of cell phones, which are ever-increasing in price. These days, companies are giving

individuals the opportunity to pay monthly for their new phones. When broken down monthly, on the surface, $20 or so may not seem unreasonable for having the latest and greatest device at your fingertips. From a consumeracy angle, though, the total price at the end of two, three, or four years is a different story. In a similar vein, car companies have started offering leases for their vehicles at a weekly rate, as opposed to a monthly rate, because smaller dollar amounts are simply more appealing to consumers. Let's not forget that we are, concurrently, being bamboozled by the mathematics of points, miles, cash back and loyalty. I think it's fair to say that it's easier to calculate the miles and points and loyalty associated with our consumer habits compared to the interest accrued on the dollars we are spending. To combat these and other examples of complicated consumerism, it would behoove those leaving school to have—in addition to literacy, numeracy, financial literacy, and other -*acy's*—a basic level of consumer numeracy.

I'll leave you with one last example.

One day, not too long ago, I stumbled upon a 40% discount on an item I had long wanted to purchase[3]. What a deal! To make me even happier, that day, the store was also offering an extra 10% off all purchases. Proudly taking the item to the counter, naturally, I had a smile on my face. The clerk behind the counter, noticing my smile, inquired about my happy state. Tongue in cheek (or so I thought), I proceeded to explain that I was super excited that the item I was purchasing was 50% off today. As she rung up the item that I handed to her, the cash register kept spitting out a number that equated to something a little less than half price (46%, to be exact). Before I could even explain the situation, a prompt manager was already punching keys on the till to override the "incorrect" discount that the machine (not *that* machine![4]) was calculating for my purchase. Already in too deep, I decided not to launch into my typical questions as to whether they had taken off the tax of the discount first. (And, no, I wasn't buying a pizza.)

> "I think it's fair to say that it's easier to calculate the miles and points and loyalty associated with our consumer habits compared to the interest accrued on the dollars we are spending."

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[3] I've decided to leave the item, the store and the dollar amount of the purchased item out of what follows.

[4] See my column in the March-April 2018 issue of *The Variable*.

Call for Contributions

The Variable is looking for contributions from all members of the mathematics education community, including classroom teachers, consultants, and teacher educators. Consider sharing a favorite lesson, an essay, a book review, or any other work of interest to mathematics teachers in Saskatchewan. If accepted for publication, your article will be shared with a wide audience of mathematics educators in Saskatchewan and beyond.

We also welcome student contributions in the form of artwork, stories, interesting problem solutions, or articles. This is a great opportunity for students to share their work with an audience beyond that of their classroom, and for teachers to recognize students' efforts during their journey of learning mathematics.

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Ilona & Nat,
Editors