The Variable

Presented by the Saskatchewan Mathematics Teachers' Society

Volume 4 Issue 1 2019

Rich Mathematical Conversations
Des conversations mathématiques riches

Purposeful Practice

Spotlight on the Profession: Dr. Glen Aikenhead

Refining Planning: Questioning with a Purpose

The Canadian Math Wars: An Abridged History

Saskatchewan Mathematics Teachers' Society

AFFILIATE
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In this issue

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SMTS Executive (2018-2020)

The Variable

Notice to Readers

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Notice to Contributors

The Variable welcomes a variety of submissions for consideration from all members of the mathematics education community, including classroom teachers, consultants, teacher educators, researchers, and students of all ages. Submitted material is assessed for interest, presentation, and suitability for the target audience.

Please submit articles by email to thevariable@smts.ca in Microsoft Word format. Authors should aim to limit submissions to 3000 words or less and include a photo and a short biographical statement of 75 words or less with their submission. Accepted contributions are subject to revision. Editors reserve the right to edit manuscripts for clarity, brevity, grammar, and style.

Editorial

It has been a busy time around the editorial table at *The Variable*. First and foremost, we would like to extend a huge thank you and congratulations to everyone who played a part in the development of *The Variable* up to this date. It is the contributors, distributors, readers, and critical friends that made us what we are today and gained us recognition as recipients of the 2019 NCTM Affiliate Publication Award. As nice as awards are, we are just as excited about the stories of teachers reading, implementing, and tinkering with the ideas contained in these pages.

Now entering our fourth year of publication, we continue to refine our medium to best reflect and support the voices of Saskatchewan teachers. This process has not been a simple one due to the overwhelming diversity of Saskatchewan classrooms, and the strength of a periodical such as this relies on the willingness of contributors to provide a lens into these varied contexts.

With this in mind, we have made some changes designed to make your reading as practical as possible in your professional lives as mathematics teachers, consultants, coaches, and enthusiasts. First, *The Variable* is introducing a new column entitled "My Favourite Lesson." It is designed for teachers to share lessons and tell stories directly from their classrooms. These stories represent the cornerstone of what we want to do: connecting teachers and amplifying their ideas. We are also moving to a semi-annual publication schedule focused on publishing issues at critical times in the professional calendar. We recognize (and live within) the ebb and flow of classroom teaching and hope that publication dates in late August and late January maximize your ability to interact with the ideas.

In this edition, we are struck by a tension often overlooked in the teaching of mathematics. Jules Bonin-Ducharme and David Earl re-introduce us to very practical pieces of classroom teaching: conversation and practice. It is here that math teachers (including ourselves) often live—in the day-to-day "battles" of building mathematical understanding. Contrast that with the synopsis Dr. Egan Chernoff provides of the political "war" of mathematics teaching. Here we see our work as a larger piece of a political milieu, one that has a substantial influence. What we are left with is this: Until now, it seems the *de facto* mantra of the classroom teacher is to focus on winning "battles" and trust that the "war" will take care of itself. However, if the teaching of mathematics sways in rhythm with political tides, this may not be a luxury that teachers are afforded much longer.

Ilona & Nat
Editors

My Favourite Lesson

The Variable exists to amplify the work of Saskatchewan teachers and to facilitate the exchange of ideas in our community of educators. We invite you to share a favorite lesson that you have created or adapted for your students and that other teachers might adapt for their own classroom. In addition to the lesson or task description, we suggest including the following:

- Curriculum connections
- Description of the lesson or task
- Anticipated student action (strategies, misconceptions, examples of student work, etc.)
- Wrap-up, next steps

To submit a lesson or if you have questions, please contact us at thevariable@smts.ca. We look forward to hearing from you!

Ilona & Nat,
Editors

Alternate Angles

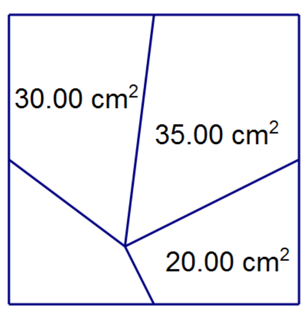
Alternate Angles is a column on problems from multiple perspectives: various methods that could be used to solve them, insights we get from their solution, the new paths that they can lead us to once they have been solved, and how they can be used in the classroom.

Exploring Areas

Shawn Godin

Welcome back and Happy New Year, problem solvers! Last issue, I left you with the following problem:

A point P is drawn inside a particular square and is joined to the midpoints of each of the sides, creating four quadrilaterals. The areas of three of the quadrilaterals are indicated in the diagram below.



Determine the area of the fourth quadrilateral.

After finding this problem online some time ago, I went back to what I was working on at the time and forgot about it, but something resonated in me about it. In its original form, the numbers were probably different, so for this column, I chose another set of numbers that worked.

So let's get to work. This problem lends itself well to exploration using dynamic geometry software. Since the three given areas add up to 85 cm^2, I will construct a 10 cm by 10 cm square to explore to see if I see any patterns. I used Geogebra to play with this problem and created Figures 1, 2 and 3. You can play with my sketch at https://ggbm.at/cy4augr3.

If we start with the point in the centre of the square, then obviously all four quadrilaterals are squares and have the same area. If we start moving P to the left, then the areas on the right (#1 and #2) increase while the ones on the left (#3 and #4) decrease. Similarly, when we move P to the right areas #1 and #2 decrease while #3 and #4 increase (see Figure 1).

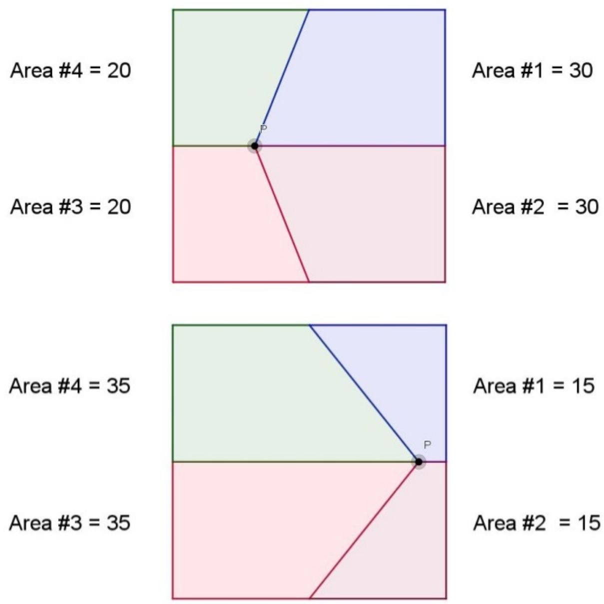


Figure 1 : Areas as P moves left and right

Moving P up and down, we notice that areas #1 and #4 decrease when P goes up and increase when P goes down, while areas #2 and #3 do the opposite (Figure 2).

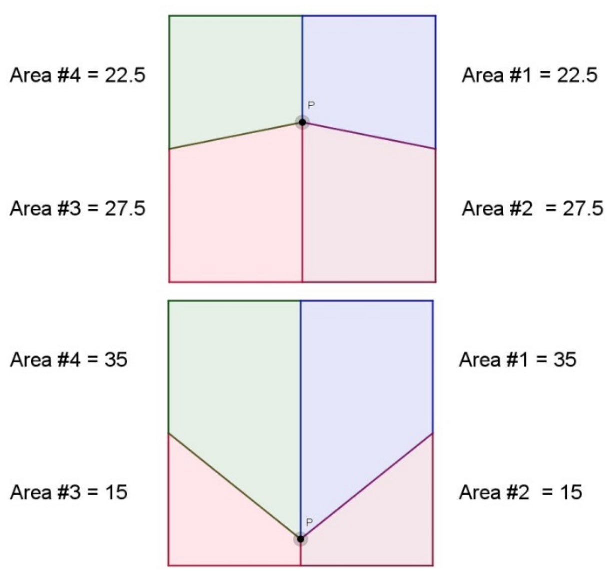


Figure 2: Areas as P moves up and down

At this point, you may have noticed that no matter how P is moved, areas #1 and #3 behave in an opposite manner, as do areas #2 and #4. That is to say, when one area is increasing, the other is decreasing. After a little more playing, it would seem that any time area #1 increases, #3 decreases by the same amount and similarly with areas #2 and #4. Thus, the sum of areas #1 and #3 is constant, as is the sum of areas #2 and #4. Notice in Figures 1, 2, and 3 that the sum of areas #1 and #3 is always 50.

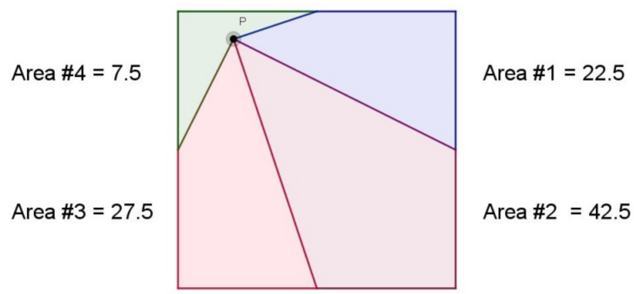


Figure 3: Area #1 + Area #3 = Area #2 + Area #4

Thus in the original problem, if we let x represent the unknown area, then we are led to

$$x + 35 = 30 + 20,$$

which leads to the unknown area being 15 cm^2.

It would be nice, however, to be able to determine this directly. Let's start by attaching a coordinate system to the square. We'll let the side length be $2s$, put opposite corners at $(0,0)$ and $(2s, 2s)$, and let the coordinates of P be (x, y), as shown in figure 4.

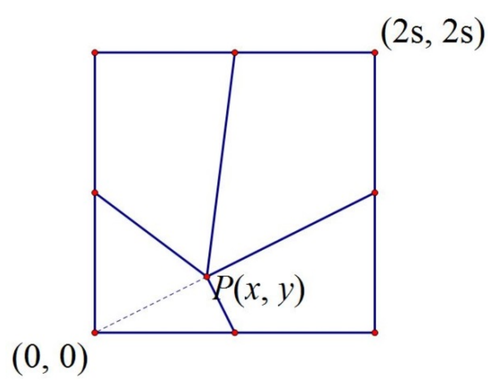


Figure 4: Attaching a coordinate system

If we add a line segment from one of the vertices of the square to P we cut it into two triangles. Each of the triangles has base s centimetres while their heights are x and y centimetres, respectively. Thus, we can represent our unknown area in terms of the variables as

$$A = \frac{s}{2}(x + y)$$

If we do the same thing to the other three quadrilaterals, we get a system of equations in our three variables:

$$\frac{s}{2}\big(x + (2s - y)\big) = 30 \qquad (1)$$
$$\frac{s}{2}\big((2s - x) + (2s - y)\big) = 35 \qquad (2)$$
$$\frac{s}{2}\big((2s - x) + y\big) = 20 \qquad (3)$$

We can simplify and solve this system and use the values of the variables to determine the desired area. The observant reader will notice that the expressions for the areas from equations (1) and (3) have the same sum as the expressions from equation (2) and the desired area (in other words, (1) + (3) = (2) + A). Thus, subtracting equation (2) from the sum of equations (1) and (3) yields

$$\frac{s}{2}(x + y) = 15,$$

our desired area, without determining the values of the variables!

As satisfying as the algebraic solution was, let's dig a bit deeper to see if we can figure out *why* the sum of the opposite quadrilaterals is invariant. If we cut all of the quadrilaterals into triangles, we will notice that there are 4 pairs of triangles with equal areas. These pairs come from creating the triangles formed by joining the vertices of the square to P and then bisecting the areas of each of these triangles by drawing the medians from P. If we label the areas of the triangles a, b, c, and d, as in Figure 5, we see that the areas of opposite quadrilaterals add up to $a + b + c + d$, exactly half the area of the square.

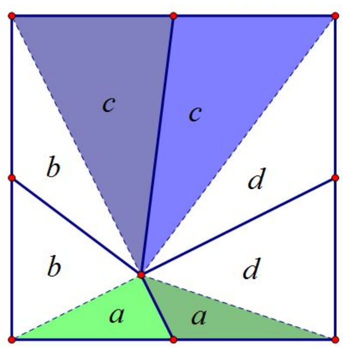


Figure 5: Cutting the quadrilaterals into triangles

So we have not only solved our problem in several different ways, we have also determined why the property that we discovered holds.

Are we done? Of course not! Many problems are only limited by our imagination. The first thing that came to my mind was, "What if we had a rectangle instead of a square?". You can have fun exploring the property we have discovered while investigating our problem and seeing for which shapes it holds.

Now, time for your homework:

Three consecutive numbers are multiplied together to form a new number, for example $7 \times 8 \times 9 = 504$. Discover all you can about numbers of this type.

Until next time, happy problem solving!

Shawn Godin teaches and is a department head at Cairine Wilson Secondary School in Orleans, Ontario. He strongly believes in the central role of problem solving in the mathematics classroom. He continues to be involved in mathematical activities: leading workshops, writing articles, working on local projects and helping create mathematics contests.

Problems to Ponder

It's a Long Way to the Top

Every time I come home, I have to climb a flight of stairs. When I'm feeling energetic I sometimes take two steps at a time. This gives me a number of ways to climb the stairs.

For example, if there are ten steps, I could climb them taking five leaps of two, giving the pattern 2, 2, 2, 2, 2. Or I could only use a leap of two at the beginning and the end, giving the pattern 2, 1, 1, 1, 1, 1, 1, 2.

How many ways are there all together of climbing the ten steps?

Extension
Being a mathematician, I don't have ten, but rather n steps. Can you find a formula to express the number of ways there are of climbing n steps using leaps of one and two?

Source: plus.maths.org/content/its-long-way-top-solution

Call for Research Grant Applications

Notice of Intent 2019-20 Funding

The McDowell Foundation is inviting PreK-12 teachers and other educators to submit a Notice of Intent to begin the grant application process for research projects funded in the 2019-20 school year. Foundation grants provide funding, guidance, and release time to support recipients' project goals and activities.

Prior research experience is not necessary. The Notice of Intent and overall application process assists applicants in developing a meaningful and achievable research project which supports professional development through reflective practice. Grants totaling $85,000 are available.

The McDowell Foundation provides research grants to explore new and innovative ways to best meet the educational needs of students in Saskatchewan. A range of topics have been funded over the past 26 years and can be reviewed on the Foundation website, www.mcdowellfoundation.ca.

Notice of Intent Requirements

A completed Notice of Intent shall include the following:

1. **Applicant(s)**
   - Project leader's name, address, phone number and email.
   - Project team members' names, phone numbers and emails.
2. **Applicant Employer**
   - School, university or other.
3. **Describe What You Intend to Study and Why**
   - Provide a 100- to 200-word summary.
4. **Research Experience**
   - New, novice, or skilled researcher.

The Notice of Intent should be emailed to mcdowell@stf.sk.ca by the November 13, 2018, deadline. The Notice of Intent can be submitted orally if the research team prefers. Please contact the Foundation for further details.

Grant Application Process

1. Submit a Notice of Intent by November 13, 2018.
2. Applicants are invited to a grant proposal development workshop in December 2018.
3. Applicants develop a draft grant proposal for review by January 28, 2019.
4. Applicants may revise the draft grant proposal based on the feedback and advice received and shall submit a final grant proposal by April 22, 2019.

McDowell FOUNDATION

In conversation with Dr. Glen Aikenhead

In this monthly column, we speak with a notable member of the mathematics education community about their work and their perspectives on the teaching and learning of mathematics. This month, we had the pleasure of speaking with Dr. Glen Aikenhead.

Glen Aikenhead is Professor Emeritus (2006), University of Saskatchewan, Canada. He earned an Honours BSc (University of Calgary, 1965), Masters of Arts in Teaching (Harvard University, 1966), and a Doctorate in Science Education (Harvard University, 1972). His research and development projects over the years emphasized relevance of school science. This interest culminated in his 2006 book Science Education for Everyday Life: Evidence Based Practice. *Since the early 1990s, Dr. Aikenhead has engaged in cross-cultural school science for both Indigenous and non-Indigenous students. In 2000, he collaboratively developed a community-based series of units* Rekindling Traditions: Cross-Cultural Science & Technology Units. *Since then he helped edit Saskatchewan's science textbook series (Grades 3-9) that includes Indigenous knowledge; he co-authored in 2011 an academic resource book* Bridging Cultures: Indigenous and Scientific Ways of Knowing Nature*; and he published a collaborative, action-research, teacher professional development program in 2014,* Enhancing School Science with Indigenous Knowledge: What We Know from Teachers and Research. *Saskatchewan's science program is well on its way to become school science for reconciliation. Currently he is involved in a mathematics and science project with Norway's Indigenous Sámi educators. His main focus, however, is the transformation of Saskatchewan's mathematics program in response to Canada's Truth and Reconciliation Commission's calls to action. The National Association for Research in Science Teaching awarded him its 2014 Distinguished Contributions to Science Education through Research Award.*

First things first, thank you for taking the time for this conversation!

When I first contacted you for an interview, I expressed my interest in learning more about your perspective on issues "within the intersection between mathematics and Indigenous culture / ways of knowing." You kindly corrected me, explaining that it would be more appropriate to refer to "the intersection between mathematics' culture / ways of knowing and Indigenous culture / ways of *knowing." Your rephrasing suggests that mathematics itself is culture-laden, imbibed with values and ideologies.*

Could you elaborate on this? In what sense is mathematics cultural, despite its popular characterization as "value-free, decontextualized, acultural, non-ideological, [and] purely objective in its use" (Aikenhead, 2017, p. 27)? What values and ideologies are embedded in Western (in particular, Euro-American) mathematics?

Let me respond first to today's popular characterization of mathematics as value-free, acultural, etc. It needs to be explained by a sequence of stories from history. This popular characterization of mathematics originated in about 400 BCE with Plato's notion of reality. He dichotomized reality into: (1) a world of ideas and (2) a world of physicality. The ancient Greek male aristocracy saw themselves as finding truth in what arises in their minds, and they saw artisans, women, and slaves as dealing with the material world, which could not be trusted to lead to any ultimate truth because one's senses could be fooled when observing this physical world. Mathematics was assigned to the world of ideas.

> "The ancient Greek male aristocracy saw themselves as finding truth in what arises in their minds, and they saw artisans, women, and slaves as dealing with the material world, which could not lead to any ultimate truth."

European cultures inherited an Arabic enhanced version of Greek mathematics during the Renaissance period (about 1300 to 1600 CE). Merchants in Renaissance times found value in using arithmetic and algebra because it helped to advance profits. Thus, mathematics in early Renaissance Europe had an everyday function.

Its scholarly function was originally closeted in monasteries, where the ancient Greek presupposition about reality continued to be held in conjunction with religious beliefs. Then later, as universities (i.e. institutions that deal with pure thought) were created across Europe, the scholarly function of mathematics began to find a home in those universities. By about 1700, mathematics' scholarly function had become a subject in elite universities such as Cambridge and Oxford. It continued to carry the implicit presupposition that mathematics belonged to Plato's world of ideas. This meant that the universe was comprised of abstract objects to be discovered only through thought processes of the (male) mind. Materialistic discoveries were ultimately not worthy unless they were associated with mathematics, which was seen as a window into God's created universe, as Sir Isaac Newton thought.

The stories from history continue with a consequence of the Industrial Revolution (1750 to 1850). It created the need for mass education, which led to the creation of public schools. In the mid-1800s, mathematics educators furiously debated which form of mathematics should be taught in public schools—the elitist version or the everyday version. Eventually the elitist Platonists won. (Details about the tactics that the Platonists used are found in Aikenhead, 2017, section 4.2.1.)

But the Platonists were not content with pushing their beliefs about mathematics into the schools; they also suppressed the everyday form of mathematics by denigrating its non-Platonist, value-laden, contextualized, ideological, and subjective features that functioned in the common people's lives (Plato's world of physicality). The non-Platonist content was considered irrelevant to the elitism valued by Platonists, and hence, unbecoming as school subject matter. This suppression was so thorough that generations after would never even contemplate teaching mathematics from the perspective of everyday culture (i.e. cultural mathematizing) or teaching how mathematics affected people's everyday lives in useful and detrimental ways; for example, artificial intelligence industries.

> "Today the Platonists' elite version of school mathematics functions as a screening device for postsecondary institutions and employers, whether or not the content taught is relevant to the program."

What has sustained this suppression, born out of Plato's dichotomy about reality? Today the Platonists' elite version of school mathematics functions as a popular screening device for postsecondary institutions and for employers, whether or not the actual content taught is relevant to the institution's program or to what mathematics is used in a specific occupation. This broad generalized screening is not valid for the purpose of identifying bright people; it identifies mathematics enthusiasts and people who can effectively memorize without much cognition. In addition, mathematics' screening causes serious social injustices.

This screening function is one of its ideologies—a term defined in general as a doctrine of how people or institutions treat others. This screening function gives mathematics its high status, which can be maintained by making the school subject purposefully difficult to learn. This ideology is about controlling what happens to people, which involves a set of values over how to control people and who to control.

The ideology of quantification is the doctrine of forcing people to quantify as many things as possible. One negative consequence for most people is the quantification of personal aspects of their lives, such as representing their intellectual understandings with a number, which usually rests on the *value* that numbers are always objective. Cultural anthropologists have studied such ideologies, naming them empiricism, rationalism, etc. A plethora of values are hidden within the Platonists' mathematics taught in schools; these are summarized in my 2017 paper (section 4.5). Mathematics is indeed a human endeavor. Plato's 400 BCE belief about reality contradicts 21st century realities.

You explain that "the term 'mathematics' functions as a superordinate *concept, representing a set of multiple mathematics knowledge systems that grammatically function as* subordinate *concepts" (2017, p. 33), each of which is inextricably connected to a historical and social context. And yet, you write that in the West, mathematics is often "not identified with its own cultural features or cultural context" (p. 133).*

How has the cultural content of Euro-American mathematics become suppressed, and what kinds of practices continue to suppress it today and perpetuate the myth of a value-free, universalist discipline? What role, if any, does the English language play?

As described just above, people in powerful positions in society tended to gain from using mathematics' high status, its purported value of objectivity, and its screening function. Any move to alter the curriculum challenges its screening function and thereby threatens to diminish the power of those who tend to gain from its screening function; usually, upper middle-class families, globalized corporations, and indirectly, political groups, for instance. Screening ensures that the natural equal distribution of talent among identifiable groups is distorted by a policy of *inequitable accessibility* that discriminates against certain races, cultures, and subcultures.

The screening function of mathematics has led to marginalizing Indigenous people from gaining high paying jobs because of low mathematics achievement. This type of systemic racism inherent in the current mathematics curriculum has been challenged by Canada's Truth and Reconciliation Commission.

Eurocentric languages do play a key role by marginalizing ideas from Indigenous cultures, which the English and French languages have no way of expressing. In other words, if we can't translate it, we ignore it. But this role of language is not as powerful as the role of Eurocentric anthropocentrism, which encourages us to believe that anything European cultures create is better than any other culture's creation. Thus, Eurocentric anthropocentrism finds ways to believe that its mathematizing is so superior that Euro-American mathematics can help colonize other races with the belief it is doing them a service. That describes today's colonial policy toward Indigenous ways of mathematizing, a policy ensconced in a doctrine of what constitutes "effective" living in today's society (a topic discussed below in answer to another question).

> "Eurocentric anthropocentrism encourages us to believe that anything European cultures create is better than any other culture's creation."

As you and many scholars argue, school mathematics curricula in North America (and in colonized nations around the world) have long privileged Euro-American mathematics at the expense of other mathematics knowledge systems, and in particular Indigenous mathematics. This has not only served to devalue Indigenous perspectives and ways of knowing and to further the process of colonization, but also to impede the achievement of Indigenous students.

Recently, an increasing number of researchers and educators in Canada have recognized this imbalance and sought to intertwine these knowledge systems. Unfortunately, some of these projects, although well-meaning, tend to tokenize or stereotype, if not further marginalize Indigenous ways of knowing. You offer the following as one example of a word problem that is ostensibly meant to bring Indigenous culture into the classroom, but actually has a distorting effect that is insulting to Indigenous people:

> Imagine a band of 250 Aboriginal People. Each tipi can hold approximately eight people. Calculate how many tipis would be needed to house the entire band. (Sterenberg, p. 21; as cited in Aikenhead, 2017, p. 15)

Could you explain why this problem, and similar problems that project Euro-American mathematics onto Indigenous contexts (e.g., using proportional reasoning in cooking bannock, or studying symmetry in traditional embroidered products), do not serve the cause of decolonization? How would you modify such a task (if it is not beyond repair)?

I'd like to answer by quoting university mathematics professor Dr. Edward Doolittle of Mohawk ancestry, who explains the effects of Euro-American projectionism. It is another example of our culture's tendency to judge other cultures' worldviews solely on the basis of our own worldview. It seems like an arrogant simplification on our part. Dr. Doolittle stated:

My feeling is that Indigenous students who are presented with such oversimplification feel that their culture has been appropriated by a powerful force for the purpose of leading them away from the culture. The [contextualized teaching materials] may be reasonable but the direction is away from the culture and toward some strange and uncomfortable place. Students may, implicitly or explicitly, come to question the motives of teachers who lead them away from the true complexities of their cultures. (2006, p. 20)

> "There needs to be reciprocity in cross-cultural learning, which goes both ways in learning the other's mathematizing."

Decolonization requires an equality of respect for, and understanding of, each culture by the people in each culture. That is a path for repairing attitudes that produce the mathematics problem you quoted.

Equality of respect and understanding has been achieved by projects in which mathematics teachers and students learn the significance and the meanings of an Indigenous mathematizing activity being explored by school mathematics students, before transforming the activity into one that fits a Euro-American context in which a Platonist idea can be taught as another culture's way of engaging in a similar situation; for example, bead work on clothing can lead eventually to teaching geometry concepts. There needs to be reciprocity in cross-cultural learning, which goes both ways in learning the other's mathematizing. The mathematics problem you quoted did not express reciprocity. It projected a mathematics ideology onto a sensitive familial Indigenous situation. It forced Indigenous students to deal with their intimately personal world in a foreign way. It was a one-way projectionist encounter by the person who wrote the offending problem.

Eurocentrism prevents many Euro-Canadians from seeing themselves as "the other." I suppose we must put ourselves in the shoes of Indigenous people in order to understand that we are "the others." This can happen the more we learn about an Indigenous worldview, which arises from learning in a deep way some of their traditional mathematizing activities found in Indigenous communities. We need to understand that teaching mathematics to most students, Indigenous and non-Indigenous, is not about turning them into mathematics enthusiasts. That is indoctrination, not education. Instead, it is about getting students to understand how the mathematics enthusiasts influence mainstream culture, and to be able to appropriate some of their powerful knowledge found in the culture of mathematics, but only the knowledge relevant to the current and future lives of those students.

You have similarly criticized the term "ethnomathematics" (and "ethnoscience"), arguing that it is a "discriminatory influence against today's decolonizing agenda for school mathematics; and as a result […] is not fully supportive of today's reconciliatory renewal of Canadian education" (p. 86). Ostensibly, the goal of ethnomathematics research is to study the mathematics knowledge systems of different cultures, to describe how cultural values influence school mathematics, and most importantly, to make school mathematics more relevant to different cultural or ethnic groups (Aikenhead, 2017, p. 56). In what ways does this work sometimes fall short of its goals?

My answer rests on the recognition that (1) school mathematics is a human/cultural phenomenon; (2) students move from their own culture into participating in the culture of school mathematics with varying degrees of difficulty (easy for math-enthusiast people, almost impossible for math-phobic people, and somewhere in between for the remaining majority); (3) the less culture clash experienced by students moving **from** their personal worldview and cultural self-identities **into** a school mathematics cultural milieu, the more successful students tend to be; and (4) D'Ambrosio and his followers present school mathematics content as being decontextualized, value-free, acultural, etc.—the Platonist position—while other cultures' or subcultures' mathematics are understood by D'Ambrosio to be contextualized, value-laden and cultural. D'Ambrosio is a world renowned pioneer in taking students' culture into consideration when teaching mathematics and science. His work began in the 1960s when he realized that poor street children in Brazil had developed their own system of arithmetic, which he learned and then figured out how to teach them Platonist mathematics by starting from the children's arithmetic system. He expanded his school and university teaching to include Brazil's Indigenous students, and he explored how Platonist mathematics has had a major role in colonization and military weapons production.

Given these four parameters, let's consider an Indigenous student in an ethnomathematics classroom, and then in a culture-based decolonized mathematics classroom.

A. A student moving **from** their Indigenous culture that is highly conscious of values, a culture that has responsibilities for maintaining those values, a culture that owes its survival to attending to contexts, and a culture that understands almost everything in terms of four aspects (emotional, physical, spiritual, and *intellectual*), **to** a culture that ethnomathematics teachers present as being value-free, no responsibilities to the answer you get except that it is the right one, decontextualized, and understands almost everything narrowly (irrelevantly?), in an *intellectual* way only.

B. A student moving **from** an Indigenous culture (described just above), **to** a mathematics culture with different values, different responsibilities for maintaining its values, contextualized in its everyday use, resulting in positive and negative effects on society, and narrowly intellectual because that's the protocol of this other culture.

In short, the difference is a move from one culture to a non-culture versus a move from one culture to another culture. For many Indigenous and non-Indigenous people, moving between two different cultures makes *more* intuitive sense than moving between a culture and a non-culture. For many people, a non-cultural context just doesn't seem real. Ethnomathematics could reduce a degree of culture clash for students by recognizing Platonist mathematics as a cultural phenomenon created by humans to better survive in their physical, social, and economic worlds, a belief ethnomathematics educators hold about other culture's mathematizing. Such a reduction in culture clash tends to facilitate greater learning of Euro-American mathematics, which is the goal of ethnomathematics educators.

> "Ethnomathematics could reduce culture clash for students by recognizing Platonist mathematics as a cultural phenomenon created by humans to survive in their physical, social, and economic worlds."

You argue that one reason that many promising curriculum renewals never reach their full potential is that they focus on inclusion *and* consultation*, rather than* dialogue *and* collaboration *with Indigenous Elders or communities (Aikenhead, 2017, p. 15). What is the difference between these two forms of engagement?*

This difference is an issue of power: who has the power to make final decisions? On the one hand, an education group can invite an Elder into a committee meeting (i.e., inclusion), listen to her or his advice (i.e., more inclusion), and then at their next meeting, with the Elder absent, make a decision. This is consultation. On the other hand (as was done during the development of the Saskatchewan science program in 2008 to 2012), an education group can be formed with Elders as partners from the beginning. Everyone engages in heart-felt discussions to understand deeply the positions of Indigenous and non-Indigenous members (i.e., a dialogue). The Elders are given the final say on matters that represent their culture, and they are given the opportunity to address issues that indirectly affect Indigenous students. It is a mutual sharing of responsibilities to understand and to decide (i.e., collaboration). Everyone equitably shares the power of decision-making. The results of collaboration tend to be much better because the decision-makers are better informed. But it takes dialogue to become better informed. Reading a book or an article cannot replace engaging in a dialogue. Cultural immersions are excellent venues for heart-felt dialogues.

In broad strokes, could you describe your vision for a decolonizing mathematics curriculum—with 'curriculum' not only in the sense of a resource that guides instruction, but also the lived, shared experience of teaching and learning mathematics? How might it benefit all *students, not only students of Indigenous background?*

> "Decolonizing mathematics classrooms can already be found in Saskatchewan where teachers are being flexible in responding to the cultural assets that students bring into their classroom."

Decolonizing mathematics classrooms can already be found in Saskatchewan where teachers are taking the time to be flexible in responding to the cultural assets that students bring into their classroom, no matter what their home culture is. Importantly, this student- and community-centred approach develops strong relationships between teachers and their students. Students' gifts and self-identities are nurtured, including those found in Indigenous cultures.

Canadians have historically tried to eliminate Indigenous cultures. Reconciliation, the chosen pathway to respond today in Canada, calls for decolonizing our schools' curricula; hence the emphasis on understanding Indigenous cultures. The teachers who are already involved in mathematics for reconciliation talk about being a learner along with their students, as well as continuing some of their normal ways of teaching. Teacher support for initiating a decolonizing classroom must begin with a cultural immersion experience of some sort, with Elders and/or knowledge holders.

Students' gifts and self-identities are diverse. For example, the mathematically gifted will be given opportunities to hone those gifts. In high schools in Saskatchewan, research has estimated that these students comprise about 26% of a provincially funded school population. For students blessed with other gifts or self-identities, understanding mathematics in terms of its function in Canadian mainstream culture is the goal, which differs from its function in specific STEM occupations into which mathematics enthusiasts, such as myself, are generally attracted.

Another goal of decolonizing mathematics for most of the 74% of high school students is creating equitable *accessibility* for students to gain excellence in adult numeracy in Canada's everyday culture. We need to remind ourselves of the adults we know who are leading very productive and happy lives in spite of their weak understanding of mathematics.

Those people represent relevant evidence when considering mathematics education in broad brush strokes.

Teachers' pedagogical flexibility depends on an uncrowded mathematics curriculum. Except for the mathematics enthusiasts, a significant amount of today's curriculum content (Grades 5-12) has been put out of date by computer programs and other technology. Students' smartphones, for instance, have replaced a significant portion of the drill and practice found in the curriculum, a curriculum that continues to help colonize Indigenous students by depressing their graduation rates.

> "Teachers' pedagogical flexibility depends on an uncrowded mathematics curriculum."

Your question "*How might it benefit* all *students?*" needs to be reworded into two questions in order to be answered accurately, because nothing in education can make a difference for "all" students. So we have to talk in terms of averages. How **does** decolonized mathematics affect students' average academic achievement? Careful systematic research into large projects that do teach decolonized mathematics shows that achievement measured by standardized mathematics tests rises dramatically for Indigenous students, and rises noticeably for non-Indigenous students. How **might** decolonized mathematics benefit most students? It could provide non-Indigenous and Indigenous students the opportunity to participate in reconciliation as envisaged by the Truth and Reconciliation Commission. That seems like a reasonable hypothesis, but it requires careful systematic research to answer that question.

Does this mean leaving certain Euro-American school mathematics content behind? According to Ball et al., who express a view shared by many educators and policymakers, "All students must have a solid grounding in mathematics to function effectively in today's world" (2005, p. 1056). Will efforts to decolonize mathematics curricula leave students unprepared for college or a career in an increasingly technological world?

First of all, statements similar to your quote from Ball et al. are never accompanied by references to systematic research into adults functioning effectively in today's world. Where are the data? Often the claim is accompanied by an example showing that some problem *could be* solved by content found in the curriculum, but not accompanied by an example of how an adult *did* solve the problem some other way.

> "For the majority of students, today's mathematics curriculum from Grades 5-12 suffers from obsolescence and systemic racism."

This is another case of mathematics educators projecting their worldview onto others who do not share a mathematical worldview. I am not claiming that students should not work hard at studying mathematics. I am claiming that, for the majority of students, today's mathematics curriculum from Grades 5-12 suffers from obsolescence and systemic racism. For example, if you google "quadratic equations in the everyday world," you'll find either (1) STEM examples relevant to mathematics enthusiasts like myself (and that is not "the everyday world"), (2) situations for which the author projected a quadratic-equation solution where other simpler solutions are feasible, or (3) made up inauthentic examples from businesses, or drew upon situations that are now outdated by computer programs or mathematics modeling technology—all for the purpose of applying content found in the mathematics curriculum but nowhere else in the everyday world.

The platitude from Ball et al. is another example of how the everyday cultural features of Platonist mathematics are suppressed by convincing (indoctrinating?) the public to believe the platitude. It clearly conveys their Platonist mathematics ideology of control. Who controls the meaning of "effectively"? (a topic mentioned above in answer to another question). Greer and Mukhopadhyay (2012) counter Ball et al.'s assertion by stating, "'Mathematics for all' has a fine equitable sound to it, but neither theoretically, nor in terms of the actual situation for students, can it bear scrutiny" (p. 240). The extent to which Ball et al.'s platitude is repeated by many educators and policymakers demonstrates the extent to which the 19th century Platonist educators have been successful at repressing what is needed for a decolonizing mathematics curriculum. Their platitude does little to support a valid policy for the 21st century. If educators and policymakers are against decolonizing the mathematics curriculum, then they are against reconciliation. That is the ethical choice for them to make.

As mentioned above, there is plenty of outdated Platonist content to be eliminated from the curriculum for the 74% of high school students whose future lies in being effectively numerate as adults to satisfy their personal needs. My 2017 paper gives a number of examples and testimonies from mathematicians who support eliminating non-essential content. The issue is a legitimate and researchable topic that requires investigative attention, not platitudes.

Also explained above, preparation for university and college programs for the mathematics enthusiasts looking forward to a future in a STEM profession is a natural part of attending to student-centered learning in a decolonizing curriculum. I envision a high school mathematics program as rigorous as a decolonizing International Baccalaureate program, with flexibility designed for students moving back and forth between it and a program designed as preparation for adult living in the everyday world of home, community, and non-STEM jobs.

Interviewed by Ilona Vashchyshyn

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Refining Planning: Questioning with a Purpose[1]

Delise R. Andrews and Karla J. Bandemer

Have you ever witnessed a really powerful mathematics discussion and wondered, "How do I make that level of discourse the norm in my mathematics classroom?" More and more teachers are taking up the call to refine the way they teach mathematics. Whether or not your state has adopted the Common Core State Standards for Mathematics (CCSSM) (Common Core State Standards Initiative, 2010), your mathematical practice standards likely include language that focuses on having students discuss their thinking and their work. Productive mathematical conversations are a significant component of developing students' mathematical understanding and ability to effectively communicate their thinking, and the types of questions we ask can either hinder or advance that development. When we have asked teachers in our district who are adept orchestrators of student discussion how they got to where they are, their response always references the need for thoughtful planning. In short, rich mathematical discourse does not happen by accident.

A wealth of literature suggests that facilitating productive mathematical conversations is critical to developing students' mathematical understandings (Chapin, O'Connor, & Anderson, 2009; National Council of Teachers of Mathematics, 2014; National Research Council, 2001; Seeley, 2016; Smith & Stein, 2011). The process is also complex and nuanced. We seek to support teachers in this work. In our professional roles, we support teams of elementary school classroom teachers as our district sharpens its focus on collaborative lesson planning for mathematics instruction. Teams of teachers who are accountable for instruction in many different subject areas often have limited time for collaborative planning of mathematics instruction. Thus, offering them strategies and supports has been crucial—those that are not only meaningful but also feasible given the significant demands on teachers' time. Toward that end, in recent years our district math department has focused professional development on the five practices for orchestrating mathematics discussions (Smith & Stein, 2011)—(1) anticipate; (2) monitor; (3) select; (4) sequence; and (5) connect)—as well as the effective Mathematics Teaching Practices discussed in *Principles to Actions: Ensuring Mathematical Success for All* (National Council of Teachers of Mathematics, 2014).

During the course of many collaborative planning sessions with teams, teachers expressed a need to make questioning more intentional as they engaged in the anticipating and monitoring phases of a task. According to Smith and Stein (2011),

> Almost all good classroom discussions begin in the same way: by inviting a student to share how he or she solved a particular problem. After the initial student response, however, classroom discussions diverge— separating into the relatively rare fruitful ones and the much more frequent unproductive show-and-tells. (p. 69)

In most classrooms, teachers ask students to share their thinking, whether at random or by thoughtfully selecting and sequencing students' ideas. We have noticed that what separates the show-and-tell discussions from the productive mathematical discussions comes down

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to what teachers know about student thinking before the whole-class discussion even begins. That knowledge comes from observations made and questions asked during students' initial work on a task. Shifting the focus from checking students' answers to analyzing students' thinking—by asking purposeful questions—can elicit critical information. Seeking to understand students' thinking while monitoring their initial work on a task better equips the teacher to purposefully select and sequence students' ideas. This, in turn, can better support meaningful mathematical connections among students' work, which provides for a more productive discussion.

The following vignettes are not representative of any one teacher or team. Rather, they are a compilation of common themes that we have observed in our work. These themes are evidence to us of the need to focus on thoughtful planning for quality questioning in mathematics instruction.

Collaborative planning vignette 1: We have anticipated; now what?
A team of teachers collaboratively plans for discussion about the following problem:

> A rectangular pool has a perimeter of 20 yards. The width of the pool is 3 yards. What is the length of the pool?

Figure 1 shows the initial notes that one teacher took during the team's planning session. The figure reflects the team members' focus on anticipating strategies for one particular problem in the lesson and misconceptions they believe that students may have about finding the perimeter of a rectangle. The planning tool used by the teacher, based on the work of Smith and Stein, details anticipated strategies and allows space for the teacher to take notes during the lesson about which students used which strategies and the order in which to discuss the strategies during the whole-class conversation.

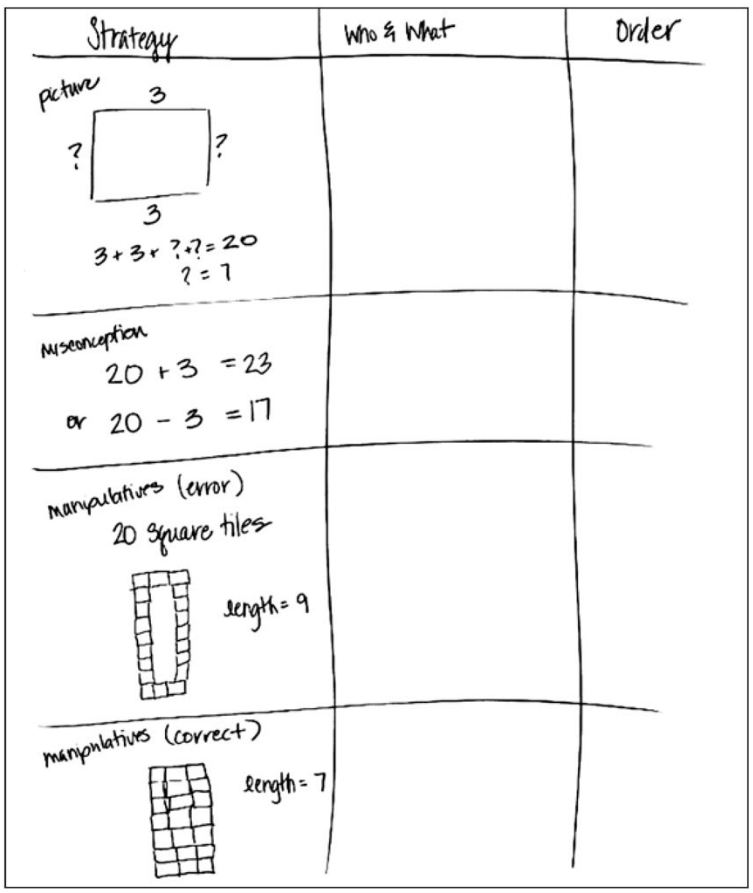

Figure 1: The teacher used a planning tool to anticipate student strategies for a particular problem, misconceptions the team believed that students might hold, and notes during the lesson about which students used which strategies and the order in which to discuss them.

On the day of the lesson, students use several of the anticipated strategies as they work on the task. Many errors are also observed that reveal potentially significant misconceptions. Eight of the eleven pairs of students in the class have incorrect answers (see Figure 2). Although the teacher and the team have anticipated some of the strategies and misconceptions, they have not planned any specific prompts or questions to elicit students' reasoning about their work. Seeing so many incorrect answers, the teacher feels somewhat overwhelmed. The source of students' confusion is unclear, so the teacher resorts to a funneling pattern of questioning (Herbel-Eisenmann & Breyfogle, 2005; Wood, 1998) to get students to the right answer. Students who have incorrect answers are asked a series of low-level, closed questions such as the following:

- What is the width of the pool?
- So, what is the width on the opposite side?
- Remember, the perimeter is the distance around the edge of the pool, so if we already have six yards, what must these other two sides add up to?

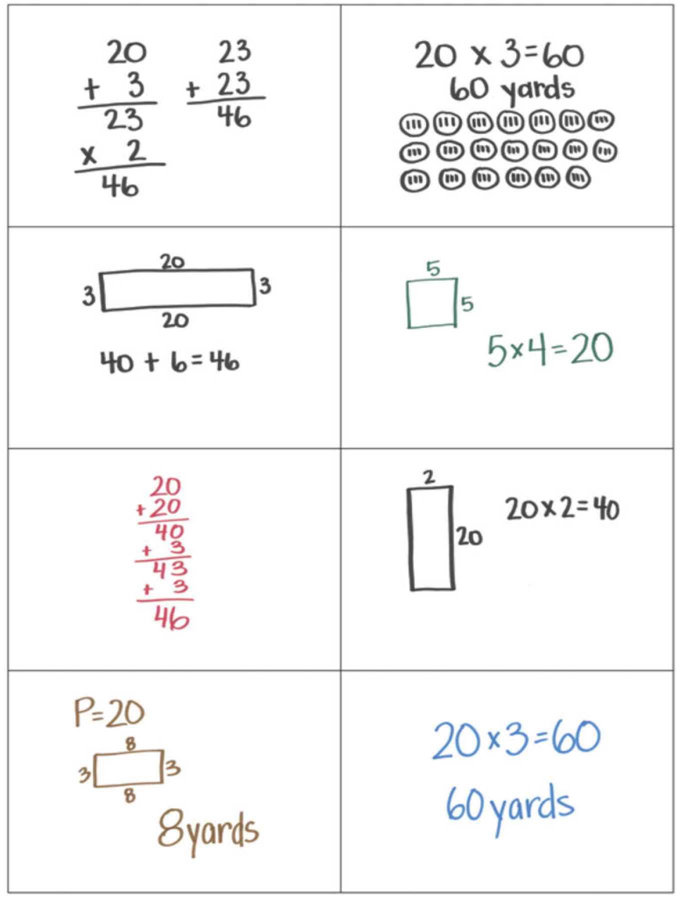

Figure 2: Although some students used anticipated strategies, many did not. The teacher resorted to questions that funneled students toward the correct answer but did not allow time to elicit and discuss students' reasoning or misconceptions.

Because the goal of the questions is to get students to the correct answer, the teacher does not feel able to walk away until the pair comes up with an answer of seven yards. Students who have correct answers are not asked questions at all but are directed to help classmates who have wrong answers. The teacher does not have enough time to touch base with every student pair.

As a result of asking mostly low-level questions, the teacher is unable to determine the nature of students' confusion (i.e., whether students do not understand the problem, are unsure of how to find the perimeter of a rectangle when given only two side lengths, are confused by the difference between area and perimeter, or have some other misconception). To begin the classroom discussion, the teacher selects a student who has a correct solution to share her work with the whole class. Students who also have the correct answer agree, and students who still do not have the correct answer erase their boards or change their work. After the lesson, the teacher expresses concern that the resulting conversation was not truly a mathematical discussion that connected strategies and deepened students' understanding about perimeter; instead it funneled students toward a correct answer. Consequently, the teacher did not address any of the misconceptions that appeared prevalent in student work. The teacher also wonders if true evidence of understanding was elicited from the students who had independently arrived at the correct answer. Is it possible that some of them had stumbled on the correct answer but still had hidden misconceptions about perimeter and area?

Although the teaching team in vignette 1 had discussed the task in advance, thinking carefully about potential student conceptions and misconceptions, the teachers had not considered how to elicit student thinking about the strategies they used. The work was valuable in that it helped the teacher feel prepared for what students might do with the task. However, the need to generate all the questions/prompts "in the moment" during instruction cost the teacher valuable time and cognitive energy and ultimately limited the ability to effectively engage with all students as they worked on the task.

Strategy	Questions/Prompts	Who/What	Order
A 202 − 198 = 004 4 more cans	• tell me about your ungrouping. • why did you subtract? • does 4 make sense? Why? • partner student with someone who used a mental math strategy - discuss connections		
B 198 + 2 = 200 200 + 2 = 202 4 cans	• Some kids subtracted... why did you decide to add? • does 4 make sense? Why? • partner with strategy C or misconception D to discuss.		
C 4 cans 2 + 2 = 4 198 200 202	• can you explain how you thought about this? • does 4 make sense? why? • if counted up, ask "would it work to count back from 202?" or vice-versa		
D misconception 202 + 198 = 400	• tell me how you thought about this problem. • does 400 make sense? Why? • what does 202 (or 198) represent in this problem? • partner w/A or B to discuss		
E error 202 − 198 = 196 196 cans	• how did you think about this problem? • does 196 make sense? Why? • would estimation help you? • could you draw a model? • partner w/ A to discuss.		
F error 202 − 198 = 114 114 cans	• explain how you thought about this problem • does 114 make sense? • tell me about your ungrouping. • partner w/A and/or E to discuss.		

Figure 3: Instead of looking for correct answers and correcting students who are struggling during the monitoring phase of the task, team members wanted to elicit and use students' thinking as a foundation for the whole-class discussion.

Collaborative planning vignette 2: Eliciting student thinking

In subsequent collaborative planning sessions, we refined our process and began to incorporate a greater focus on eliciting students' thinking. Given limited planning time and the shifts in practice that we were asking teachers to make, we initially focused on reorienting teacher prompts (from getting answers to seeking information) during the monitoring phase of the lesson. In the following vignette, a teaching team plans for discussion about the following problem:

> The local food bank needs 202 cans of beans for the food drive. So far, they have collected 198 cans of beans. How many cans does the food bank still need?

As the team plans, they anticipate possible strategies and misconceptions (see Figure 3). Instead of simply giving corrections to students who are struggling and looking for correct answers during the monitoring phase of the task, team members want to elicit and use students' thinking as a foundation for the whole-class discussion. As they talk together about how they can shift from answer-getting strategies to information-seeking methods, they plan some general open-ended prompts they will use to elicit students' thinking during this phase. Prompts written for successful strategies are designed to press students' reasoning, uncover potential confusions, and extend their thinking.

Informal conversations about previous lessons had uncovered a common theme of students neglecting to consider the reasonableness of their answers. As a result, one of the goals for this class discussion is to have students talk about whether different answers make sense. With that in mind, the team plans to listen carefully to how students support their thinking in response to planned prompts (e.g., "Does 400 make sense? Why?"). The team decides to select and sequence student work with a goal of engaging the whole class in a discussion about reasonableness. That is, the teachers plan to look for students to share one correct solution and one or two examples of unreasonable results during the class discussion.

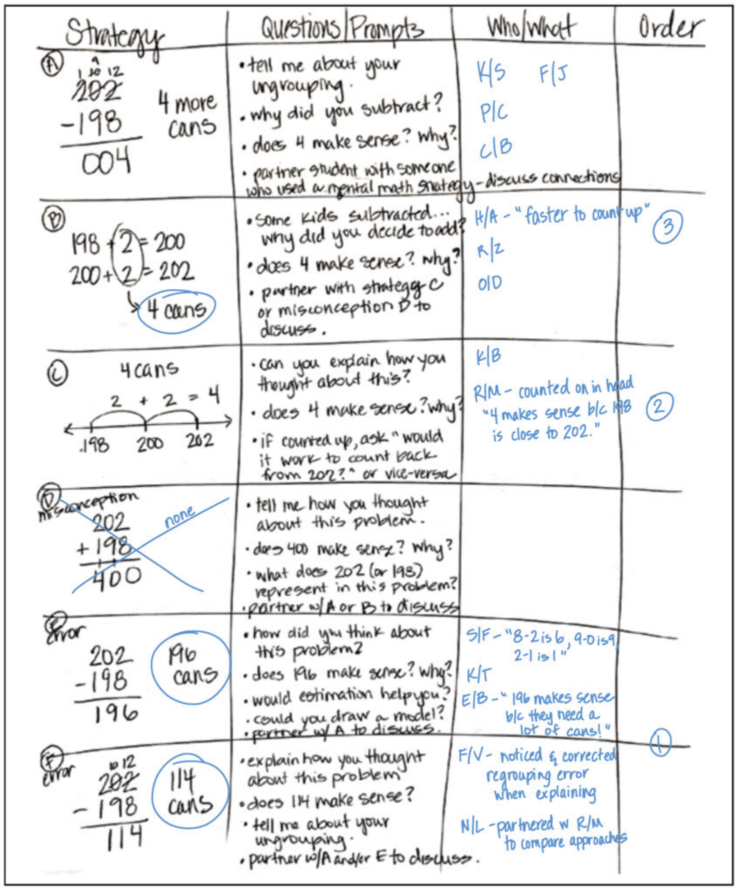

Figure 4: This teacher used the planned prompts as a guide when interacting with students, took notes after students shared their thinking, discovered two of the three anticipated incorrect answers, and asked the class to consider the answers' reasonableness within small groups before engaging in a whole-class discussion.

In one classroom on the day of the lesson, the teacher uses the planned prompts as a guide for interacting with students who are working on the task. She takes notes after students share their thinking (see Figure 4); student initials indicate which pairs of students used which strategy. The teacher notices that several students used a counting-on strategy to find the correct answer. Two of the three anticipated incorrect answers are also discovered in student work. The teacher writes the three most common answers on the board: 4 more cans, 196 more cans, and 114 more cans. Students are asked to consider the reasonableness of the three answers within their small groups before the class comes back together for a whole-class discussion.

Prepared with prompts that the team collaborated on for the monitoring phase of the lesson, the teacher was able to devote more energy during that time to thinking about students' thinking. These prompts also served to hold the teacher accountable to listen to students' ideas instead of simply focusing on whether they had the right answer. The teacher felt that with the shift in the purpose of the discussion, more students could meaningfully engage in the conversation, so she challenged the class to determine which answer made the most sense and then pushed students to support their own thinking and consider the thinking of others.

In this second vignette, the teacher also worked with a collaborative team to plan the task in advance, thinking carefully about potential student conceptions and misconceptions. The team developed purposeful questions and prompts as an added part of their planning process. Being intentional about planning questions that connected to the anticipated student strategies minimized the tendency to fall into a funneling pattern of questions. Instead, the teacher was prepared both for the solutions that students offered and how to respond in a way that helped elicit more information about students' thinking. This, in turn, provided the substance for a whole-group discussion (Smith & Stein, 2011).

The teachers and teams we have had the opportunity to work with have inspired us to think deeply about the role of planning for questioning in the development of a powerful mathematics discussion. The work of anticipating students' strategies, conceptions, and misconceptions, and the planning of questions has helped teachers move away from a showand- tell discussion and toward intentional discourse. NCTM's 2017 Taking Action series (e.g., Huinker & Bill, 2017) includes planning templates that can support teachers who are seeking to incorporate a purposeful focus on questioning as they plan for mathematics instruction and student discourse.

A call to action: Examining beliefs

Just as the teachers we work with have inspired us to refine our planning practices, we hope that this article encourages readers to reflect on how they prepare for meaningful mathematical discussions. Wood and Hackett (2017) remind us that "the substantial learning outcomes for everyone in the classroom (teachers and students alike) make it worth the time investment of using purposeful questions" (p. 58). We challenge readers to examine their own beliefs about the planning process, its purpose, and the potential impact it can have on student learning. Reflecting honestly on their beliefs about students' capacity to contribute to a mathematics discussion is also important for teachers (and instructional leaders). These beliefs play a key role in either supporting or impeding the work of planning for mathematics discourse. See Table 1 for a summary of some productive and unproductive beliefs we have encountered in our work (and sometimes needed to overcome in our own thinking).

We believe that it is possible to make productive mathematics discussions the norm in every classroom. Student thinking is the foundation for meaningful conversations that advance learning. Gaining the necessary insight into students' thinking can be done by asking purposeful questions during the monitoring phase of a lesson. In our work with teachers, we have focused on shifting the purpose of teachers' questions away from getting students to the correct answers and toward understanding students' conceptions and misconceptions about the mathematics within a task. This shift better equips teachers to orchestrate powerful mathematics discussions. By gaining an understanding of students' thinking before a discussion, teachers can select work to be shared and discussed not on the basis of how correct the answer is, but rather because of how the reasoning behind it can be used to build students' understanding about the concept. Students are empowered to defend their reasoning about a task and position themselves as mathematicians in their own right.

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Rich Mathematical Conversations

Jules Bonin-Ducharme

I've noticed that I often ask the question, "Who is talking?" when I teach students or observe teachers in my role as math facilitator. Too often, I see students listening and teachers talking for most of the lesson. Is there an exchange of views and opinions? Is there conversation?

Conversations in the math classroom are important. Concepts tend to stick more when we have opportunities to talk about them, and conversation has played an important role in learning and discovery since ancient times. Consider, for instance, Pythagoras' followers. These followers were divided into two factions, which mirrored the two aspects of Pythagorean learning. The *Akousmatikoi* (from the word *akousmata*, meaning "things heard") followed the teachings of Pythagoras. They listened and lived his philosophy without questioning it. The *Mathematikoi* (from the word *mathemata*, meaning "things studied or learned"), on the other hand, had conversations about philosophy, music, astronomy, and mathematics. They constructed new ideas in all of these branches of study on the Pythagorean belief that everything was related to numbers. In fact, the word *mathematics* is said to have its root in the word *mathemata*.

While listening is also an important part of learning, conversation pushes the learning further because it involves an interchange of ideas. Often, conversations help students discern connections that would have remained hidden if they had only listened to one idea. But how can we foster rich conversations in our math classrooms, especially if we rarely experienced conversations in the classroom as students?

Modeling Classroom Conversations

The following model (Figure 1) represents typical interactions in a math classroom. Mathematical ideas are transmitted to the teacher, who transmits them to the students. The arrows in the diagram represent the direction of this transmission. The students cannot have a conversation about the mathematics, because the teacher controls their access to the ideas and only lets students see his or her own version of them.

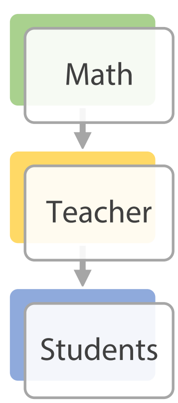


Figure 1

Another pattern that I often see in classrooms is sometimes, perhaps wrongly, referred to as "differentiation." In this case, the math is transmitted individually to each student by the teacher based on perceived needs. As seen in the model, the teacher is still seen as a translator and distributor of mathematics. This pattern is modelled in Figure 2.

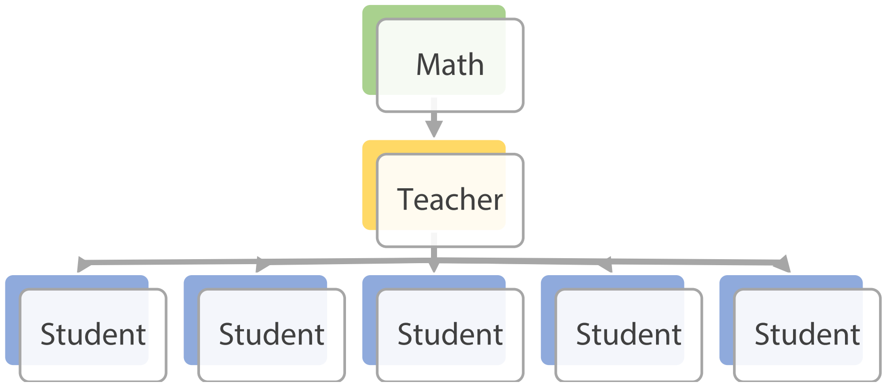


Figure 2

Group work has also been suggested as an efficient teaching practice that has more potential to foster conversations among students. However, even when students are working together, it is possible for the teacher to maintain their status as the translator and distributor of mathematics, as the model below (Figure 3) suggests. Here, students might have conversations about the teacher's interpretation of the mathematics, but still do not access the information in a direct way.

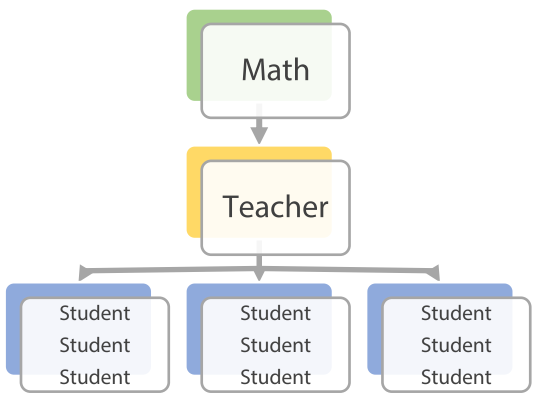


Figure 3

In my view, if we want students to engage in rich meaningful mathematical conversations, we must provide them direct access to the mathematics. This also means that the classroom experiences a role reversal where the teachers now must interpret mathematics through their students' perspective. In this model (Figure 4), teachers have the role of a facilitator, rather than a gatekeeper.

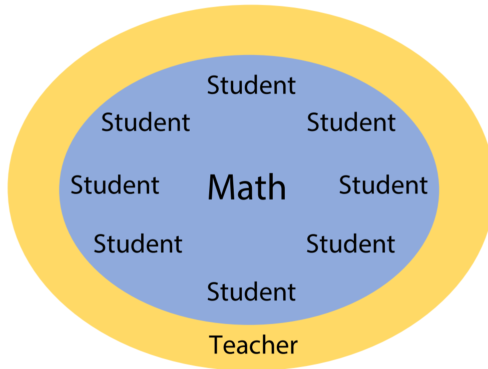


Figure 4

Good Conversations Start with Good Tasks

While teachers need to provide space for the conversation, the task must also be suited to conversation. The challenge for the teacher, then, is to provide opportunities for these meaningful conversations to occur.

This means providing tasks that ask students to do more than execute a sequence of strict expectations that are set out by the teacher in succession. A task should encourage students to make conjectures, argue and justify their reasonableness, and then execute their strategy to move toward a solution. This does not mean that students have full control of the direction of the lesson, which would not necessarily lead to meaningful learning. The teacher must have clear goals in mind and an intention for every learning activity.

The book *5 Practices for Orchestrating Productive Mathematics Discussions* (NCTM) includes a table (page 16) that provides a general list of characteristics of low-level and high-level mathematical tasks. I find this structure helpful when deciding if a task might encourage rich conversation.

Lower-level demands	Higher-level demands
Memorization	Procedures with connections
Procedures without connections	Thinking and reasoning mathematically

I have taken the liberty of renaming the bottom right section from "Doing mathematics" to "Thinking and reasoning mathematically." I think that this better reflects the intention of these types of higher-level tasks. To illustrate how a variety of tasks might introduce the same outcome with different opportunities for conversation, I offer a series of sample tasks that a teacher might use while working with students on the surface area and volume of a cylinder.

Memorization

Example 1: What is the formula for the area of a circle?

This type of prompt is not likely to spark much conversation in class. At most, there might be a discussion if a student mistakenly gives the formula for the perimeter $C = \pi d$ or $C = 2\pi r$ instead of the formula for the area $A = \pi r^2$. Although memorization-type prompts can be used, we must realise that they "have no connections to the concepts or meaning that underlies the facts, rules, formulas or definitions being learned" (Smith & Stein, 2011, p. 16).

Procedures without connections

Example 2: What is the area of a circle of radius 5? Hint: The area of a circle is $A = \pi r^2$.

Students will apply the formula to determine the area as a procedure. The focus is on "producing the correct answer instead of developing mathematical understanding" (Smith & Stein, 2011, p. 16). This type of prompt also doesn't offer much opportunity for conversation in class. At most, there might be a discussion if a student makes a mistake in the procedure, such as multiplying the radius by 2 instead of squaring it.

Procedures with connections

Example 3

Circle A: Area = 226 cm^2 Circle B: Circumference = 66 cm

Which circle is bigger?

This is a shift here from the previous lower-level questions. In tasks that join procedures with connections to concepts, students "engage with conceptual ideas that underlie the procedures to complete the task successfully and that develop understanding" (Smith & Stein, 2011, p. 16).

Even in reading the above question, students will likely find themselves thinking about the formulas for the area and the circumference of a circle. Then, before the students begin to solve the problem, there is likely to be a discussion about what it means for a circle to be bigger. At this point, teachers must resist simply giving an explanation about what makes a circle bigger. Allowing the students to construct their own definition will create a richer conversation. To solve the problem, students will again apply the formula for the area and the circumference as a procedure. The difference is that now, they have an opportunity to make connections between both formulas.

It's interesting to note that some students might compare the areas, others the circumferences, and still others the radii of the two circles. A rich conversation can be had by asking about the differences in comparing areas, circumferences, and radii, during which students will be able to make connections between these three concepts.

Thinking and reasoning mathematically

Example 4

Cylinder A: Volume = 480 cm^3 Cylinder B: Surface area = 480 cm^2

Which cylinder is bigger? What would the cylinders look like?

While seemingly simple at first glance, this question requires lots of thinking and reasoning. What do we mean by a "big" cylinder? What makes it bigger than the other? If the volume is to be maintained, what happens to the height as the radius gets bigger? What happens to the radius as the height grows bigger? Is the effect the same to maintain the surface area?[1]

[1] Students can use digital tool such as Geogebra to visualize the differences between both cylinders. I have built an interactive model that can be accessed at https://ggbm.at/hntwyntb

Such tasks, which demand thinking and reasoning, "require students to explore and understand the nature of the mathematical concepts, processes or relationships" (Smith & Stein, 2011, p. 16).

Volume = 480 cm^3
Cylinder A

Surface area = 480 cm^2
Cylinder B

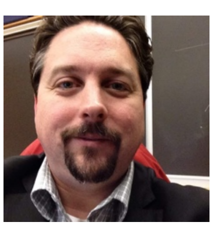
The more thinking and reasoning is offered in a question, the richer the surrounding mathematical conversation will be. However, for the conversation to occur, we have to make sure we know what our intention for the question or task is. The follow-up questions we ask students are also important. "Without specific questions that make the connections between the different strategies and the intention of the lesson, the lesson would become a show-and-tell and the link to key ideas would be lost" (Smith & Stein, 2011, p. 50).

Conclusion

If we want to foster rich mathematical conversations in our classrooms, we have to plan for it. This plan starts with the recognition that opportunities for conversations are maximized when the teachers provides direct access to mathematics through tasks that provide space for thinking and reasoning to emerge. So, while planning a lesson for the next day, instead of thinking: "What will I make my students do tomorrow?", consider: "What do I want my students to think about? How am I going to facilitate that thinking?"

References

Smith, M. S., & Stein, M. K. (2011). *5 practices for orchestrating productive mathematics discussions.* Reston, VA: National Council of Teachers of Mathematics.



Jules Bonin-Ducharme has taught mathematics, drama and computer science in high school since 2006. On October 3, 2012, he received a certificate of honor for the Prime Minister of Canada's Award for Teaching Excellence. In 2013, he became a Grade 7-10 math facilitator for the Grandes-Rivières Catholic School Board. From 2014-2018, he worked as a Grade 7-12 provincial math facilitator for the CFORP. He is now returning to the Grandes-Rivières Catholic School Board as a math facilitator from Kindergarten to Grade 12.

Des conversations mathématiques riches

Jules Bonin-Ducharme

Je remarque que je me pose souvent la question « Qui parle? » quand j'enseigne ou quand j'observe des enseignantes ou des enseignants dans mon rôle de facilitateur en mathématiques. Trop souvent, je vois des élèves qui écoutent et des enseignantes ou des enseignants qui parlent pendant la majeure partie de la leçon. Y a-t-il un échange d'idées ou d'opinions? Y a-t-il une conversation?

Les conversations en classe de mathématiques sont importantes. Les concepts ont tendance à coller lorsque nous avons l'occasion de parler d'eux, et la conversation a joué un rôle important dans l'apprentissage et la découverte depuis l'Antiquité. Considérons, par exemple, les adeptes de Pythagore. Ces adeptes étaient divisés en deux factions, qui reflétaient les deux aspects de l'apprentissage de Pythagore. Les *Akousmatikoi* (du mot *akousmata*, qui signifie «choses entendues») suivaient les enseignements de Pythagore. Ils ont écouté et vécu sa philosophie sans la remettre en question. Les *Mathematikoi* (du mot *mathemata*, qui signifie «choses étudiées ou apprises»), par contre, ont eu des conversations sur la philosophie, la musique, l'astronomie et les mathématiques. Ils ont construit de nouvelles idées dans toutes ces branches d'étude sur la croyance pythagoricienne selon laquelle tout était lié aux nombres. En fait, on dit que le mot *mathématique* a sa racine dans le mot *mathemata*.

Bien que l'écoute soit également un élément important de l'apprentissage, la conversation le pousse plus loin, car elle implique un échange d'idées. Souvent, les conversations aident les élèves à discerner les liens qui seraient restés cachés s'ils n'avaient écouté qu'une seule idée. Mais comment pouvons-nous favoriser des conversations riches dans nos classes de mathématiques, en particulier si nous avons rarement eu des conversations en classe en tant qu'élèves?

Modélisation de conversations en salle de classe

Le modèle suivant (Figure 1) représente les interactions typiques dans une classe de mathématiques. Les idées mathématiques sont transmises à l'enseignante ou à l'enseignant, qui les transmet aux élèves. Les flèches dans le diagramme représentent le sens de cette transmission. Les élèves ne peuvent pas avoir de conversation sur les mathématiques, car l'enseignante ou l'enseignant contrôle leur accès aux idées et ne les laisse voir que sa propre version.

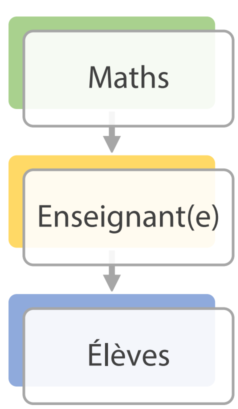


Figure 1

Un autre modèle que je vois souvent dans les salles de classe est parfois, peut-être à tort, appelé « différenciation ». Dans ce cas, les mathématiques sont transmises individuellement à chaque élève par l'enseignante ou l'enseignant en fonction des besoins perçus. Comme le montre le modèle, l'enseignante ou l'enseignant est toujours considéré comme un traducteur et un distributeur de mathématiques. Ce modèle est représenté à la figure 2.

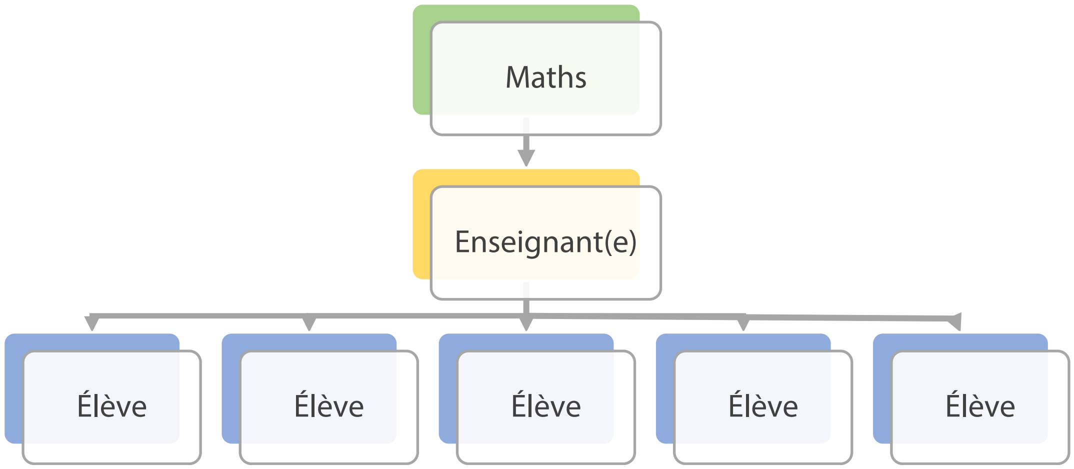


Figure 2

Le travail en petit groupe a également été suggéré comme une pratique d'enseignement efficace, susceptible de favoriser les échanges entre élèves. Cependant, même lorsque les élèves travaillent ensemble, l'enseignante ou l'enseignant peut conserver son statut de traducteur et distributeur de mathématiques, comme le suggère le modèle ci-dessous (Figure 3). Ici, les élèves peuvent discuter de l'interprétation des mathématiques par l'enseignante ou l'enseignant, mais n'ont toujours pas accès à l'information de manière directe.

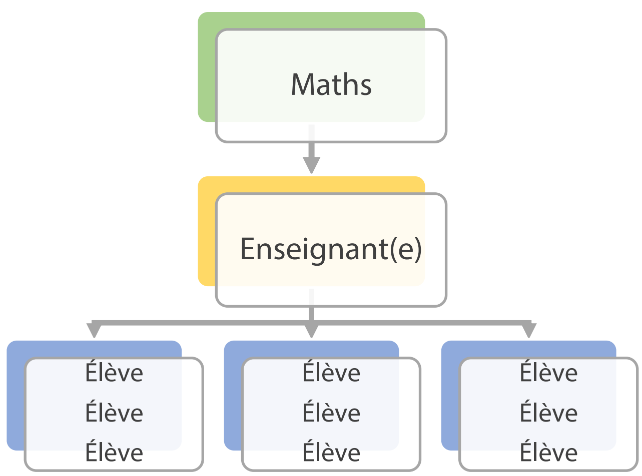


Figure 3

À mon avis, si nous voulons que les élèves s'engagent dans des conversations mathématiques riches et enrichissantes, nous devons leur fournir un accès direct aux mathématiques. Cela signifie également que la classe subit un renversement des rôles et que les enseignantes et les enseignants doivent maintenant interpréter les mathématiques selon la perspective de leurs élèves. Dans ce modèle (figure 4), les enseignantes et les enseignants jouent le rôle de facilitateur plutôt que de gardien.

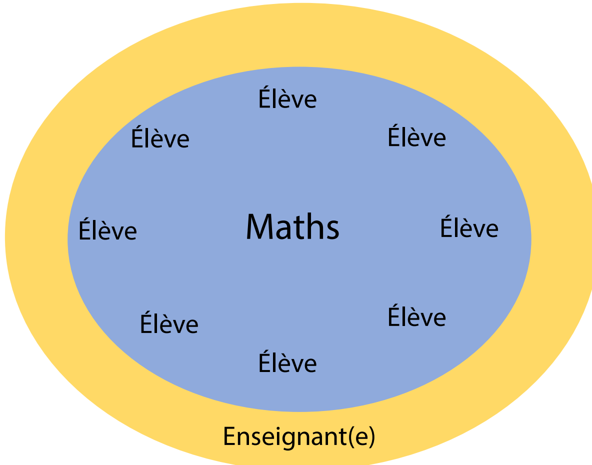


Figure 4

De bonnes conversations commencent par de bonnes tâches
Bien que les enseignantes et les enseignants doivent fournir un espace-temps pour la conversation, la tâche doit également mener à la conversation. Le défi pour l'enseignante ou l'enseignant consiste donc à donner l'occasion à ces conversations riches de se dérouler.

Cela implique de fournir des tâches qui demandent aux élèves de faire plus que d'exécuter une séquence d'attentes strictes, définies par l'enseignante ou l'enseignant l'une à la suite des autres. Une tâche devrait encourager les élèves à formuler des conjectures, à argumenter et à en justifier leur caractère raisonnable, puis à exécuter leur stratégie pour progresser vers une solution. Cela ne signifie pas que les élèves ont le plein contrôle de la direction de la leçon, ce qui ne mènerait pas nécessairement à un apprentissage significatif. L'enseignante ou l'enseignant doit avoir des objectifs clairs et une intention pour chaque activité d'apprentissage.

Le livre *5 Practices for Orchestrating Productive Mathematics Discussions* (NCTM) comprend un tableau (page 16) qui fournit une liste générale des caractéristiques des tâches mathématiques de bas et de haut niveau. Je trouve cette structure utile pour décider si une tâche peut encourager une conversation riche.

Demande cognitive inférieure	Demande cognitive supérieure
Mémorisation	Procédures avec connexions
Procédures sans connexions	Penser et raisonner mathématiquement

J'ai pris la liberté de renommer la partie inférieure droite de «Faire des mathématiques» en

« Penser et raisonner de mathématiquement ». Je pense que cela reflète mieux l'intention de ces types de tâches de niveau supérieur. Pour illustrer le fait que diverses tâches peuvent générer le même résultat avec différentes opportunités de conversation, je propose une série d'exemples de tâches que l'enseignante ou l'enseignant peut utiliser lorsqu'il travaille avec des élèves sur l'aire de surface et le volume d'un cylindre.

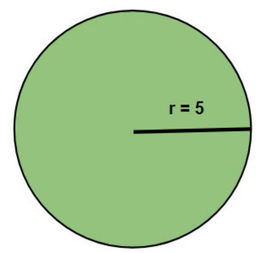
Mémorisation

Exemple 1: Quelle est la formule pour l'aire d'un cercle?

Ce type de question ne risque pas d'engendrer beaucoup de conversation en classe. Tout au plus pourrait-il y avoir une discussion si un élève donne par erreur la formule du périmètre $C = \pi d$ ou $C = 2\pi r$ au lieu de la formule pour l'aire $A = \pi r^2$. Bien que des questions de type mémorisation puissent être utilisées, nous devons nous rendre compte qu'elles « n'ont aucun lien avec les concepts ou la signification qui sous-tendent les faits, règles, formules ou définitions apprises » (Smith & Stein, 2011, p. 16).

Procédures sans connexions

Exemple 2: Quelle est l'aire d'un cercle de rayon 5? Indice: L'aire d'un cercle est $A = \pi r^2$.



Les élèves appliqueront la formule pour déterminer l'aire en tant que procédure. L'objectif est de «produire la bonne réponse au lieu de développer la compréhension mathématique» (Smith & Stein, 2011, p. 16). Ce type de question n'offre pas non plus beaucoup d'occasions de conversation en classe. Tout au plus pourrait-il y avoir une discussion si un élève commettait une erreur dans la procédure, par exemple en multipliant le rayon par 2 au lieu de l'élevé au carré.

Procédures avec connexions

Exemple 3

Cercle A : Aire = 226 cm^2 Cercle B : Circonférence = 66 cm

Quel cercle est le plus grand ?

C'est un changement par rapport aux questions de niveau inférieur précédentes. Dans les tâches qui joignent les procédures aux connexions avec les concepts, les élèves «abordent des idées conceptuelles qui sous-tendent les procédures pour mener à bien la tâche et développent la compréhension» (Smith & Stein, 2011, p. 16).

Même en lisant la question ci-dessus, les élèves vont probablement se retrouver en train de réfléchir aux formules pour l'aire et la circonférence d'un cercle. Ensuite, avant que les élèves ne commencent à résoudre le problème, il y aura probablement une discussion sur ce que cela signifie pour un cercle d'être plus grand. À ce stade, les enseignantes et les enseignants doivent résister à la simple explication de ce qui rend un cercle plus grand. Permettre aux élèves de construire leur propre définition créera une conversation plus riche. Pour résoudre le problème, les élèves appliqueront à nouveau la formule de l'aire et de la circonférence en guise de procédure. La différence est que maintenant, ils ont la possibilité de faire des liens entre les deux formules.

Il est intéressant de noter que certains élèves peuvent comparer les aires, d'autres les circonférences et d'autres encore les rayons des deux cercles. Vous pouvez engager une conversation enrichissante en interrogeant les élèves sur les différences de comparaison d'aires, de circonférences et de rayons, au cours desquelles les élèves seront en mesure d'établir des liens entre ces trois concepts.

Penser et raisonner mathématiquement

Exemple 4

Cylindre A : Volume = 480 cm^3 Cylindre B : Aire de la surface = 480 cm^2

Quel cylindre est le plus gros? À quoi ressemblent les cylindres?

Bien qu'apparemment simple au premier abord, cette question nécessite beaucoup de réflexion et de raisonnement. Qu'entendons-nous par 'gros' cylindre? Qu'est-ce qui le rend plus gros que l'autre? Si le volume doit être maintenu, qu'advient-il de la hauteur lorsque le rayon augmente? Qu'advient-il du rayon lorsque la hauteur devient plus grande? L'effet est-il le même de maintenir l'aire de la surface?[2] De telles tâches, qui nécessitent une réflexion et un raisonnement, « obligent les étudiants à explorer et à comprendre la nature des concepts, des processus ou des relations mathématiques » (Smith & Stein, 2011, p. 16).

Volume = 480 cm^3
Cylindre A

Aire de surface = 480 cm^2
Cylindre B

Plus une question offre à l'élève de penser et de raisonner, plus la conversation mathématique environnante sera riche. Cependant, pour que la conversation se produise, nous devons nous assurer de connaître notre intention concernant la question ou la tâche. Les questions de suivis que nous posons aux élèves sont également importantes. «Sans questions spécifiques faisant le lien entre les différentes stratégies et l'intention de la leçon,

[2] Les élèves peuvent utiliser des outils technologiques tels que Géogebra afin de visualiser les différences entre les deux cylindres. J'ai construit un modèle interactif accessible à ce lien : https://ggbm.at/hntwynfb

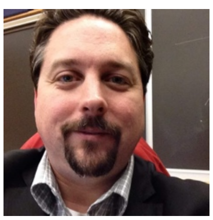
la leçon deviendrait une démonstration et le lien entre les idées clés serait perdu» (Smith & Stein, 2011, p. 50).

Conclusion

Si nous voulons favoriser des conversations mathématiques riches dans nos classes, nous devons le planifier. Ce plan commence par la reconnaissance du fait que les possibilités de conversation sont maximisées lorsque les enseignantes et les enseignants offrent un accès direct aux mathématiques par le biais de tâches offrant un espace de réflexion et de raisonnement. Donc, tout en planifiant une leçon pour le lendemain, au lieu de penser: « Qu'est-ce que je ferai faire à mes élèves demain? », considérez: « À quoi est-ce que je veux que mes élèves pensent? Comment vais-je faciliter cette réflexion? »

Références

Smith, M. S., & Stein, M. K. (2011). *5 practices for orchestrating productive mathematics discussions.* Reston, VA: National Council of Teachers of Mathematics.



Jules Bonin-Ducharme enseigne les « tiques » (les mathématiques, arts dramatiques et informatiques) au secondaire depuis 2006. Le 3 octobre 2012, il est récipiendaire d'un certificat d'honneur pour le prix du Premier Ministre du Canada pour l'excellence en enseignement. En 2013, il devient conseiller pédagogique en numératie 7e à la 10e année au Conseil scolaire catholique des Grandes-Rivières. Il a œuvré de 2014-2018 comme conseiller pédagogique en numératie de la 7e à la 12e année, volet appui aux initiatives ministérielles pour le CFORP. Il est maintenant de retour au Conseil scolaire catholique des Grandes-Rivières comme facilitateur systémique en numératie de la Maternelle à la 12e année.

Problems to Ponder

A Christmas tree is made by stacking successively smaller cones. The largest cone has a base of radius 1 unit and a height of 2 units. Each smaller cone has a radius 3/4 of the previous cone and a height 3/4 of the previous cone. Its base overlaps the previous cone, sitting at a height 3/4 of the way up the previous cone.

What are the dimensions of the smallest cone, by volume, that will contain the whole tree for any number of cones?

Source: http://mei.org.uk/month-item-18

Purposeful Practice

Dave Earl

"Learning isn't linear, it's recursive."
–Hattie et al., 2017

Recently, I've been thinking a lot about how to create purposeful practice opportunities for young math students. I draw on my experience as a basketball coach, where I've always believed in the importance of fundamental skill development, regardless of the ages of the athletes. To me, a portion of practice time should always be allocated to developing and maintaining critical foundational skills, as well as correcting any errors that players are currently making. Of course, opportunities for the athletes to apply their skills in various ways should also be provided, given that a player's capacity to apply foundational skills to the larger aspects of team play is critical in team success.

I find it odd that I wasn't taught mathematics with a similar approach. Instead, I remember practice in mathematics looking almost invariably like this:

Today's homework: p. 100 #1-29 odd.

Look familiar? As a young math learner, I was frequently asked to solve multiple exercises of the same or similar type with the hope that I would "master" the concept. More often than not, this was the only opportunity I was offered to engage with the concept—that is, until I was expected to solve similar exercises on an exam. While there was sometimes a progression from rote procedure to applications, all too often, this type of assignment was dominated by exercises that never required a higher level of thinking on the part of the student (or the teacher!). The equivalent experience in basketball would be an athlete taking fifty shots from a single location (and never returning to this location during practice again), never having to receive a pass, never having to worry about footwork, and certainly not taking the shots at game speed or in a game situation.

> "Is it possible to be more targeted and strategic about how practice opportunities are created for students when we have large class sizes and limited time?"

But the truth is that early in my career, it was common for me to assign similar assignments to my students. However, when I began to teach secondary mathematics, I began to ask myself: "Is it possible to be more targeted and strategic about how practice opportunities are created for students when we have large class sizes and limited time?" Today, my response begins with a few questions of my own: "Can the approach described earlier, where students are asked to solve long, disconnected sets of routine exercises, meet the needs of *all* learners? Does it help those students who struggle, or does it only turn them further away from mathematics?" Ultimately, I see two main issues with this approach to practice for math students: First, it doesn't allow students to continue to practice a new skill after the corresponding homework assignment has been completed, and second, it doesn't offer students the opportunity to practice in meaningful ways. Purposeful practice is achieved when both of these issues have been addressed. From a practical perspective, I see purposeful practice as built on a foundation of three characteristics: spaced practice, feedback, and student choice.

The Foundations of Purposeful Practice

Spaced practice

According to Hattie et al. (2017), “spaced practice” (practice opportunities provided throughout the duration of a course), as opposed to “mass practice” (many questions given at one time), is a more effective way of moving conceptual understanding into long term memory—or, in other words, of achieving what we may call *mastery*. Steve Leinwand supports this stance:

> Almost no one masters something new after one or two lessons and one or two homework assignments. That is why one of the most effective strategies for fostering mastery and retention of critical skills is daily, cumulative review at the beginning of every lesson. (Leinwand, 2015)

With spaced practice, students are more likely to remember what they have learned in the long term and less likely to require re-teaching of concepts they have previously seen.

Daily cumulative review correlates well with my experiences with fundamental skill development in sport, where fundamental skills need to be consistently practiced not only in isolation, but in simulated game situations. From a practical standpoint, although it may seem daunting, spaced practice is quite manageable—all that is required is a few minutes per day. More importantly, spaced practice is purposeful, because concepts are addressed and revisited over time in a strategic manner, allowing the teacher to be responsive to student needs while at the same time providing *all* students with opportunities to further develop their fundamental skills. Spaced practice thus satisfies the first characteristic of purposeful practice: allowing students to continue to review and apply important concepts and skills over the course of a year or a semester.

> “A critical aspect of providing meaningful practice opportunities is feedback, without which students may not be able to progress in their learning.”

Feedback

A critical aspect of providing meaningful practice opportunities is feedback, without which students may not be able to progress in their learning. Is the student on the right track? Are the student’s errors procedural or conceptual? Effective feedback can also help to establish a learning alliance between the teacher and the student that validates students’ learning efforts and makes them feel supported even when they make mistakes. Such feedback acknowledges work of high quality but also identifies errors and provides students guidance in correcting them.

Tyler Friesen, a colleague who teaches at Centennial Collegiate in Saskatoon, employs a simple, but effective coding system that he used to inform his students about the nature of their errors. A star meant that the error was conceptual and that they should have a discussion with him in order to resolve the issue. An asterisk, on the other hand, let the student know that the mistake was procedural and that they should:

1. Determine their error and correct it, or:

2. See a classmate if they needed support in correcting the error, or:
3. See him if the above two strategies failed.

Critical to this process was that students received assessments back in a timely manner (usually the next day) and that they always had the opportunity, in class (usually in small groups), to correct their work. Students kept their assessments and used them as a learning tool, in addition to a measure of their progress.

Student choice

Providing students with a choice when it comes to their practice can make this practice more effective, engaging, and meaningful, and can be initiated at an early grade level. Tomlinson (2000) identifies choice as a powerful way to differentiate for students with diverse learning needs. Splitting a set of practice tasks into several blocks, for example, is one simple but effective way to provide student choice. Such a practice set may look something like this:

> **Block 1**: These tasks (usually around 5) are the "must do's." These tasks range in complexity and cognitive demand but are used to demonstrate depth of understanding of a concept. Typically, these are the exercises that I take in and provide feedback on.

If students struggle with these tasks and feel that they need to first practice the required foundational skills, they move to the "Block 2" set before they complete the "must do" problems in Block 1.

> **Block 2**: Students work on this set to develop the foundational skills required to do Block 1 tasks. In this set, I usually provide many exercises for students to practice; however, students are given the choice as to how many they complete.

If the Block 1 tasks have been completed and students are interested in going deeper, they are given the option to further apply their understanding in the Block 3 set.

> **Block 3**: These tasks are designed to challenge students by increasing the cognitive load and provoking productive struggle.

This strategy is practical, in that it is not overwhelming from a "marking" perspective. It's formative, in that a great deal is learned about students based on their choices. And it's meaningful, as it promotes deep learning for students by providing control over their own differentiation.

Classroom Examples of Purposeful Practice

When practice opportunities include quality feedback and student choice, and where skill development and conceptual understanding are developed over time, the practice will surely be purposeful. However, having taught high school mathematics for over 20 years, I still have to ask questions about practicality. In other words, how could I possible do this on a daily basis? Luckily, in my role as consultant, I have had the opportunity to work with many teachers in diverse settings and to observe purposeful practice at work across grade levels with students with a variety of learning needs. In what follows, I offer some examples of purposeful practice.

2-4-2 Homework

Steve Leinwand promotes the idea that a student's daily homework should follow a 2-4-2 model (Leinwand, n.d.):

- 2 questions that practice the new skill;
- 4 questions that provide a cumulative review, comprising of a question from the day before, a question from the week before, a question from the month before, and possibly one question that is a diagnostic assessment related to the next concept to be discussed; and
- 2 questions that require problem solving, reasoning, and justification.

Figure 5 gives an example of a 2-4-2 homework assignment[3] created by Sarah Myers, a Grade 8 teacher from Saskatoon Public Schools. Her students were working on the concepts of surface area and volume the day that she gave her students the following assignment to complete. They had previously done work with rational numbers.

1. What is the surface area of a cube with side length 4.6 cm?
2. What is the volume of a pool with these dimensions: length of 50 m, width of 25 m, and depth of 3 m?
3. 0.19÷0.152 (Hint: What do you have to do to "get rid of" the decimal in the divisor first?)
4. What pattern do you see? Write the next 3 numbers in the pattern.

   1, 1 ¾, 2 ½, _______, _________, __________
5. 4/7 – ?/? = 5/21 Show your work.
6. What number times 0.078 gives a product of 7.8? How do you know?
7. What is 74% of 322?
8. Riddle: I am a nine-digit number. I contain each digit from 1 to 9 except the digit 8, and I contain two appearances of the digit 5. Discover what number I am by using the following clues:
   - I am less than 500 000 000.
   - My ten millions digit and my ones digit are the same.
   - The sum of my hundred millions, ten millions, and millions digits is 18.
   - My thousands digit is 1.
   - My ten thousands digit is one more than my hundred thousands digit.
   - My ones digit is equal to the sum of my hundreds digit and my tens digit.
   - My hundreds digit is 3.

   I AM THIS NUMBER: ___ ___ ___ ___ ___ ___ ___ ___ ___

Figure 5: A 2-4-2 homework assignment.

[3] For more examples of Leinwand's 2-4-2 homework assignments, head to his website: http://steveleinwand.com/presentations-2/

The question I immediately asked when I was first introduced to this approach to homework was, "Are there enough questions for kids to really learn the new concepts?" Well, for some, maybe not, which would require the teacher to respond accordingly. In their formative assessment, teachers can identify the students who need extra support and practice with the new concepts and skills. For most students, however, under the condition that the new concept is reviewed cumulatively over the duration of a course, Leinwand's homework strategy will be sufficient.

Row games

If you would like to assign multiple exercises to give all of your students the opportunity to practice a new procedure or problem type, then consider doing so with a row game. A row game, introduced to the math education community by Kate Nowak (2009), is a simple but powerful strategy in which students work in pairs on what looks like a typical work sheet. However, this particular worksheet has a hidden power! Typically, these worksheets are either organized into two columns, or two separate worksheets are prepared, one for each member of the pair. Although the questions in each column, or on each worksheet, are different, corresponding questions generate the same answer. If each member of the pair is working on the first question from their set and obtain the same answer, they know that they've (almost certainly) arrived at a correct result. But the real magic happens when the same result is *not* achieved, as the pair must then work together to find and correct the error(s). This is the hidden power of row games: They are self-checking, and they give students an opportunity to learn with and from one another.

Figure 6 gives an example of a typical row game, in this case targeting solving two-step linear equations. (I've only included a couple questions in each column to save some space. Typically, row game worksheets include around 6-10 questions per column.)

Show your work.	
Partner A	Partner B
$2x - 3 = 5$	$3x - 5 = 7$
$5c + 4 = 49$	$3c + 7 = 34$

Figure 6: Excerpt of a row game

Row games are purposeful for many reasons, beyond providing multiple practice questions: they create discourse, provoke metacognition, and deepen understanding through error analysis.[4]

Cumulative review

Beginning a class with a 5- to 10-minute warm up activity on a skill that is applicable throughout the grades is another way of providing ongoing purposeful practice for students. For example, operations with integers and fractions, and factoring numbers and

[4] For access to dozens of pre-created row game worksheets that target multiple content areas, head to the following website: https://app.box.com/v/rowgames

polynomial expressions are worth revisiting multiple times throughout the duration of a course. In a joint effort between Saskatoon Public Schools and the Greater Saskatoon Catholic School Division, a series of cumulative review worksheets are being developed for each grade. For Grade 9, the daily reviews, comprised of four questions each, focus on adding, subtracting, multiplying, and dividing whole numbers, integers, fractions, and decimals. *Figure 7* gives an example of a 4-day cycle, where each operation is applied to each content area once per week (you'll notice that the operations remain the same while the content moves one spot per day). Some teachers have implemented daily cumulative review by having their students complete the questions individually on mini whiteboards and then correct their work with a partner. Each week, this cycle repeats itself with new questions.

Monday:	Tuesday:	Wednesday:	Thursday:
Find the Sum: $23 + 15$	Find the Sum: $7.9 + 4.4$	Find the Sum: $\frac{3}{5} + \frac{1}{3}$	Find the Sum: $(-17) + (-5)$
Find the Difference: $(-3) - (-8)$	Find the Difference: $22 - 14$	Find the Difference: $2.8 - 1.9$	Find the Difference: $\left(-\frac{4}{5}\right) + \frac{1}{2}$
Find the Product: $\left(\frac{5}{6}\right)\left(-\frac{9}{8}\right)$	Find the Product: $(-5)(9)$	Find the Product: 21×50	Find the Product: 3.5×7.2
Find the Quotient: $7 \div 7.1$	Find the Quotient: $3\frac{1}{6} \div 2\frac{3}{8}$	Find the Quotient: $(-24) \div (-8)$	Find the Quotient: $35 \div 7$

Figure 7: An example of a cumulative review progression

This strategy is also in line with the philosophy of outcome-based assessment, as it provides a mechanism for giving students multiple opportunities to demonstrate that they have reached an outcome. Further, this concept could reduce the amount of marking required by a teacher, given that much of the work is formative, and as such could be assessed by the students themselves, or possibly by peers.

Steven Vincent, a high school teacher at Tommy Douglas in Saskatoon, describes how he incorporates cumulative review into his instruction:

> Once a week (I picked Fridays), I give students a one-page handout with questions from each unit that we have covered to date. They complete it in the first 15 minutes of class while I do a homework check or catch up with individual students. I post an answer key at the back of the room and they are responsible for marking their own sheets. I don't usually take them in. Every second Friday, I give them the option of handing in their sheet if they think they have now reached a higher level of understanding than they demonstrated on a unit exam. They need to do very well

> (4/4 questions correct or 3/4 with only a computational error) in order for me to justify an improvement in their mark. I explain that these sheets can't hurt them in terms of lowering their marks.
>
> This week, I decided to do a more comprehensive cumulative review and divided the course into four sections: factoring, irrational numbers, graphing, and solving systems. I made four separate sheets from Kuta Software and had my students choose two of the four sheets to complete and hand in at the end of the week. Then, this Friday, I am going to give them an option to complete and hand in their work in one out of four of the areas. Essentially, this works out to an opportunity every two weeks to demonstrate their improved understanding of one outcome. Sometimes it is 2 or 3 outcomes, but usually just one so as to control my workload.

Steve's strategy is practical as it answers the question, "How do we do this with such limited time?" It is also powerful in the way that it provides spaced practice to his students and the opportunity to improve and strengthen their understanding over the entire length of a course.

Instructional routines

Instructional routines are another way to provide purposeful practice for our students while taking into account time restrictions and large class sizes. In Saskatoon Public Schools, we have chosen to focus on two instructional strategies in primary and middle years: Number Talks and Hands-on Games and Activities. These strategies were chosen for multiple reasons, but first and foremost because they generate high-yield results with respect to improving students' computational fluency. Further, they are purposeful, as both are used strategically to respond to student need and to deepen conceptual understanding.

Number Talks (Humphreys & Parker, 2015; Parish, 2010) usually target mental math, but can be used in many different ways as an instructional tool. Often, a number talk begins with teachers posing a problem and having their students quietly ponder it without pencil and paper. Once a student has a solution, they are instructed to notify the teacher by giving a "thumbs up." While the teacher allows other students more time to consider the question, those who already have a thumb up are encouraged to find another way to solve the problem. If they find another way, they are instructed to put up a second finger, and so on… The "thumb up" (and possibly additional fingers) signals that the student is prepared to share not only their answer with the class, but also their reasoning. While the student explains, the teacher carefully records the student's explanation on the board for the rest of the class to see. As the teacher calls on multiple students, multiple strategies become visible. As an example, *Figure 8* shows three possible strategies for mentally multiplying 9 by 18:

Solve 9×18 mentally.

A. Use **friendly numbers**.	B. Use **partial products**.
$10 \times 18 = 180$ $180 - 18 = 162$	$9 \times 10 = 90$ $9 \times 8 = 72$ $90 + 72 = 162$

C. **Break a factor** into smaller factors.

9×18
$9 \times (9 \times 2)$
$81 \times 2 = 162$

Figure 8: Three mental multiplication strategies

Through number talks, students quickly discover that there are multiple ways to solve problems, some more efficient than others, but in all cases the opportunity to explain their own thinking and hear others explanations is powerful. Moreover, the strategy is flexible and adaptable to your students' needs. For example, while Number Talks typically focus on developing mental math skills, in some cases it may be appropriate to let students use mini whiteboards during a Number Talk to help organize their thinking or to represent a solution in different ways.[5]

> "Through number talks, students quickly discover that there are multiple ways to solve problems. Moreover, the strategy is flexible and adaptable to your students' needs."

Used strategically, hands-on games and activities are another great way to help students deepen their understanding and develop computational fluency. There are literally hundreds of options out there, so I will highlight just one of my favorites: (Min + Max)imize.[6] (Min + Max)imize is a game I was introduced to by Nat Banting, a high school teacher in our school division. To play the game, students are arranged in groups of 3 or 4, and ideally have a large whiteboard to record their thinking. The game begins with the teacher rolling a set of 4 dice and telling the students the values that were rolled. The teacher also provides the students with the operations they are to incorporate in the mathematical expression they are about to create. Each groups of students is required to create and evaluate a mathematical expression using the numbers and operations given to them, with the goal of either maximizing or minimizing the expression. Greatest value (or least, if the goal is to minimize) wins! This game is easily differentiable to meet the needs and level of the students playing, and encourages mathematical discourse, metacognition, problem solving, and collaboration.

Summary

While purposeful practice can be defined in many ways, I see it as the combination of spaced opportunities, responsive instruction, feedback, and student choice. It is a commitment to provide practice to students in thoughtful, well planned, strategic, and therefore meaningful ways. This is similar, in many respects, to how an expert coach would construct his basketball seasons, with many opportunities to practice skills in a variety of

[5] For more information about Number Talks, see *Making Number Talks Matter* by C. Humphreys and R. Parker (2015). You can also find a summary in the following blog post by Catherine Reed: http://brownbagteacher.com/number-talks-how-and-why/

[6] For "(Min + Max) imize" see a complete explanation plus differentiation ideas on Nat Banting's blog here. A similar version of this game is called "Beat the Teacher." For more information about this and other hands-on games and activities, head to the National Council of Teachers of Mathematics website (www.nctm.org) and search "games."

contexts. Time and large class sizes do not need to be barriers, as there are practical ways to make this happen through instructional and formative assessment strategies. It's time to eliminate the one-size-fits-all, "#1-29 odd" approach to practice that simply does not meet the diverse needs of all of our students.

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David has been an educator in Saskatchewan for over 25 years. He has been fortunate to hold many roles during his career including teacher, coach, consultant, and, most recently, high school administrator. David is a passionate teacher and a lover of mathematics, and is a strong believer that all students can learn given the right time and supports. David is married to wife Jodi, has two daughters Taylor and Lauren, and currently is the Acting Vice Principal at Walter Murray Collegiate in Saskatoon.

In this column, you'll find information about upcoming math education-related workshops, conferences, and other events. Some events fill up fast, so don't delay signing up!

For more information about a particular event or to register, follow the link provided below the description. If you know about an event that should be on our list, please contact us at *thevariable@smts.ca*.

Within Saskatchewan

Technology in Mathematics Foundations and Pre-Calculus
March 7, 2019
Regina, SK
Presented by the Saskatchewan Professional Development Unit

Technology is a tool that allows students to understand senior mathematics in a deeper way. This workshop is designed to have math foundations and pre-calculus teachers experience a variety of technology tools that allow students to represent and visualize mathematics concepts. Tools highlighted are useful for students to explore, learn, communicate, collaborate and practice, in order to enhance their understanding of mathematics in secondary mathematics.
More information at https://www.stf.sk.ca/professional-resources/events-calendar/technology-mathematics-foundations-and-pre-calculus

Accreditation Renewal/Second Seminar
March 7, April 12, 2019
Saskatoon, SK
Presented by the Saskatchewan Professional Development Unit

Accreditation seminars are offered to enable qualified teachers to become accredited. Accreditation is the process by which qualified teachers are granted the responsibility of determining the final mark or standing of the students in a specified Grade 12 (level 30) subject or subjects.

More information at https://www.stf.sk.ca/professional-resources/events-calendar/accreditation-renewalsecond-seminar-0

Making Math Class Work
April 11, 2019
Meadow Lake, SK
Presented by the Saskatchewan Professional Development Unit

Math classrooms across Saskatchewan are increasingly complex and diverse. Meeting everyone's needs can be daunting, even with all of the instructional strategies and structures available to teachers. Number Talks, Guided Math, Rich Tasks, Problem Based Learning, Open Questions, High Yield Routines are just some of the strategies available to teachers, but where to start? Come work collaboratively to problem solve how to make math class work for you and your students.

More information at https://www.stf.sk.ca/professional-resources/events-calendar/making-math-class-work-0

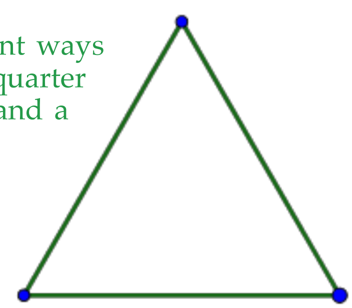
Early Learning With Block Play – Numeracy, Science, Literacy and So Much More!
April 12, 2019
Moose Jaw, SK
Presented by the Saskatchewan Professional Development Unit

This is a one-day workshop for early learning educators from prekindergarten, kindergarten and Grade 1 to work collaboratively to discover and deepen their understandings around the many foundational skills that children develop during block play. Through concrete, hands-on activities, participants will experience and examine the many connections between block play and curricular outcomes, and the current research on the topic.

More information at https://www.stf.sk.ca/professional-resources/events-calendar/early-learning-block-play-numeracy-science-literacy-more

Problems to Ponder

Given an equilateral triangle, in how many different ways can you construct a shape that has an area that is a quarter of the original triangle, using just a straight edge and a pair of compasses?



Source: http://mei.org.uk/month-item-18

Beyond Saskatchewan

NCTM Annual Meeting and Exposition
April 3-6, 2019
San Diego, CA
Presented by the National Council of Teachers of Mathematics

Join thousands of your mathematics education peers at the premier math education event of the year! Network and exchange ideas, engage with innovation in the field, and discover new learning practices that will drive student success. Classroom teachers, administrators, math coaches, supervisors, college professors, and preservice teachers can all benefit from the sessions and learning at this event.

Head to http://www.nctm.org/Conferences-and-Professional-Development/Annual-Meeting-and-Exposition/

46th Annual OAME Conference: All In
May 16-18, 201
Ottawa, ON
Presented by the Ontario Association for Mathematics Education Conference
Join hundreds of your mathematics education peers in Ottawa, Ontario for the 46th Annual OAME Conference. This year's featured speakers include Marian Small, Eli Luberoff, Jules Bonin-Ducharme, Tracy Zager, Nat Banting, and many more!

For more information, head to https://oame2019.ca

Online Workshops

Education Week Math Webinars
Once a month, Education Weekly has a webinar focusing on math. They also host their previous webinars on this site. Previous webinars include Formative Assessment, Dynamic vs. Static Assessment, Productive Struggling, and Differentiation.

Past webinars: www.edweek.org/ew/webinars/math-webinars.html
Upcoming webinars: www.edweek.org/ew/marketplace/webinars/webinars.html

Did you know that the SMTS is a **National Council of Teachers of Mathematics Affiliate**? NCTM members enjoy discounts on resources and professional development opportunities, access to professional journals, and more. When registering for an NCTM membership, support the SMTS by noting your affiliation during registration.

NCTM | AFFILIATE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

Tangents

Extracurricular Opportunities for K-12 Students

This column highlights local and national extracurricular opportunities for K-12 students interested in mathematics, including collaborative, individual, online, and in-person challenges, contests, and camps. For dates, registration procedures and applications, and other information about the contests listed, please head to the contest websites, included below the descriptions. If we have missed an event that should be on our list, please contact us at thevariable@smts.ca.

If you are looking for contests available in other provinces, head to the Canadian Mathematical Society website (cms.math.ca/Competitions/othercanadian). The CMS also maintains a list of resources for students who are looking to build their problem-solving skills and succeed in competitions: see cms.math.ca/Competitions/problemsolving.

Canadian Math Kangaroo Contest
March 24, 2019

The purpose of this competition is to introduce youngsters from Grade 1 to Grade 12 to math challenges in a fun and enjoyable way, thus inspiring their further interest and advancement in mathematics. The competition is held yearly in more than 40 Canadian cities. Students may choose to participate in English or in French.

More information at kangaroo.math.ca/index.php?lang=en

Caribou Mathematics Competition
Held six times throughout the school year

The Caribou Mathematics Competition is a worldwide online contest that is held six times throughout the school year. Each of these days, five contests are offered, one for each of the grade levels 3/4, 5/6, 7/8, 9/10 and 11/12 and each one in English, French and Persian. Available in English, French, and Persian.

More information at caribouteststs.com

Euclid Mathematics Contest
Written in April
Presented by the Centre for Education in Mathematics and Computing (University of Waterloo)

The Euclid Mathematics Contest is an opportunity for students in their final year of secondary school and motivated students in lower grades to have fun and to develop their mathematical problem solving ability. Most of the problems are based on the mathematical curriculum up to and including the final year of secondary school. Most of the problems are based on curricula up to and including the final year of secondary school. Some content might require students to extend their knowledge and the best way to familiarize oneself with commonly appearing topics is to practice using past contests.

More information at www.cemc.uwaterloo.ca/contests/euclid.html

Fryer, Galois, and Hypatia Mathematics Contests
Written in April
Presented by the Centre for Education in Mathematics and Computing (University of Waterloo)

The Fryer, Galois and Hypatia Math Contests are an opportunity for students to write a full-solution contest. They are fun way to develop mathematical problem solving skills through a written mathematical activity. For students in Grades 9 (Fryer), 10 (Galois) and 11 (Hypatia). Questions are based on curriculum common to all Canadian provinces. Rather than testing content, most of the contest problems test logical thinking and mathematical problem solving.

More information at www.cemc.uwaterloo.ca/contests/fgh.html

Gauss Mathematics Contests
Written in May
Presented by the Centre for Education in Mathematics and Computing (University of Waterloo)

The Gauss Contests are an opportunity for students to have fun and to develop their mathematical problem solving ability. For all students in Grades 7 and 8 and interested students from lower grades. Questions are based on curriculum common to all Canadian provinces.

More information at www.cemc.uwaterloo.ca/contests/gauss.html

Opti-Math
Written in March
Presented by the Groupe des responsables en mathématique au secondaire

A French-language mathematics challenge for secondary students who would like to exercise and develop their problem-solving skills.

More information at www.optimath.ca/index.html

Pascal, Cayley, and Fermat Contests
Written in February
Presented by the Centre for Education in Mathematics and Computing (University of Waterloo)

The Pascal, Cayley and Fermat Contests are an opportunity for students to have fun and to develop their mathematical problem solving ability. For students in Grades 9 (Fryer), 10 (Galois) and 11 (Hypatia). Early questions require only concepts found in the curriculum common to all provinces. The last few questions are designed to test ingenuity and insight. Rather than testing content, most of the contest problems test logical thinking and mathematical problem solving.

More information at www.cemc.uwaterloo.ca/contests/pcf.html

The Virtual Mathematical Marathon
Supported by the Canadian National Science and Engineering Research Council

The virtual Mathematical Marathon has been developed by an international team of mathematicians, mathematics educators and computer science specialists with the help of the Canadian National Science and Engineering Research Council and its PromoScience program.

The main activity is a competition allowing students to enjoy solving challenging mathematical problems all year around. Students can join the game at any time and at no cost, simply creating an individual profile with an individual username and a password. Available in French.

More information at www8.umoncton.ca/umcm-mmv/index.php

Problems to Ponder

Twenty Divided Into Six

Katie had a pack of twenty cards numbered from 1 to 20.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20

She arranged the cards into six piles. The numbers on the cards in each pile added to the same total.

What was the total, and how could this be done?

Source: https://nrich.maths.org/1047

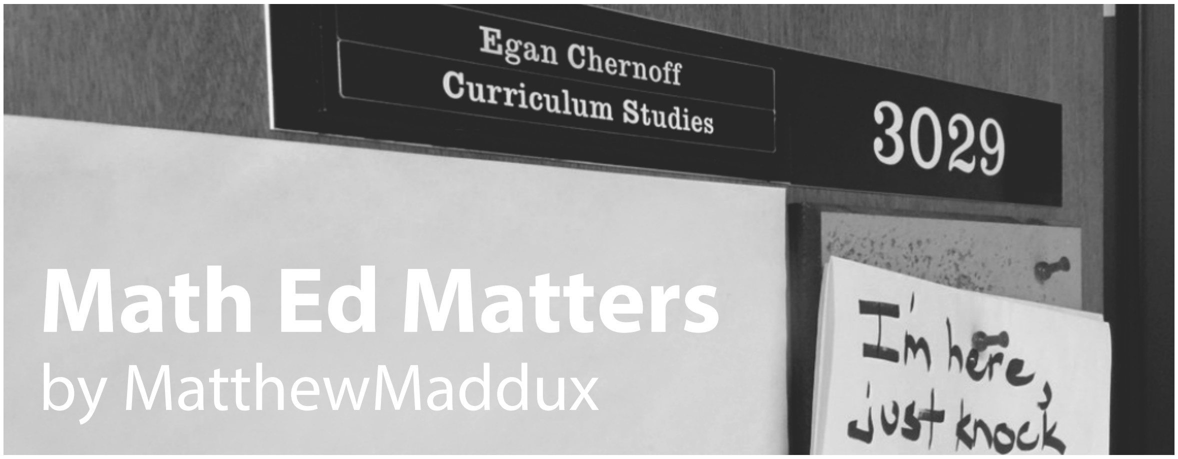
Odd at Heart

The digits 1-9 can be arranged in a square so that the digits in no column, row, or diagonal appear in order of magnitude. For example,

2	8	7
6	1	4
5	3	9

Show that, however this is done, the central digit must be odd.

Source: Pi Mu Epsilon Journal 5(4), 209. Available at http://www.pme-math.org/journal/issues/



Math Ed Matters by MatthewMaddux *is a column telling slightly bent, untold, true stories of mathematics teaching and learning.*

The Canadian Math Wars: An Abridged History

Egan J Chernoff
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According to some, the election of Donald Trump as the 45th President of the United States is the natural culmination of the Republican Party embracing the Tea Party. Did you know that a somewhat similar story is playing out here in Canada? Douglas Robert Ford's recent anointment, not only as the 26th Premier of Ontario, but as the *de facto* leader of the back-to-basics movement in mathematics education, I contend, is the natural culmination of the Canadian Math Wars (a disagreement over how mathematics is and should be taught in schools) being subsumed by the Canadian Culture Wars (disagreements between Conservatives and Liberals). Let's see if you agree.

If you've been following the news recently, especially in Ontario during the months of September and October, it might appear that the recent debate over how mathematics should best be learned and taught in schools stems from mathematics students in Ontario not doing so well during their latest efforts at meeting provincial math standards. Reading the multiple recent takes on the issue—say, for example, from Margaret Wente and Caroline Alphonso, as found in *The Globe and Mail*, or other individuals writing for publications such as *The Star*—may have confirmed the recency of this debate for you. It should be noted, though, that the (most recent) disagreement over the teaching and learning of mathematics has been taking place for approximately 7 years. As part of my job, I've followed this debate on a daily basis since September 2011. So, if you'll indulge me, I'll do my best to provide an abridged historical overview of the last 7 years of The Canadian Math Wars, and how we got to where we are today.

On September 21, 2011, Michael Zwaagstra, working for the Frontier Centre for Public Policy (FCPP), published a report entitled "Math Instruction that Makes Sense: Defending Traditional Instruction." (It should be noted, the title and accompanying picture on the report was clearly meant as a shot across the bow of one of the most widely adopted textbooks series at the time, *Math Makes Sense*.) In what I deem a clever move by the FCPP, they released the study to the media via an FCPP Commentary entitled, "New Math is

Failing our Students: Teaching techniques leave students without a solid foundation." The use of the term *new math* was clever because it immediately equated any current changes taking place in the math classroom with the failed, dramatic overhaul of the way mathematics was taught that took place in the 1960s—referred to as the "new math." Use of this particular term immediately had discerning parents (especially those who had either heard about or perhaps even experienced first-hand the "horrors" of the new math) questioning what was currently happening in their child's math classroom. There was no way they would let their precious son or daughter succumb to the horrors that they had experienced or heard about! The media ran with the new math story, and, as you'll soon see, it had legs.

> "Use of the term *new math* immediately had discerning parents questioning what was happening in their child's math classroom."

Here in the prairies, the story of new math became, well, *de rigeuer*. Soon, it became a recurring topic of heated discussions in newspapers (e.g., *Winnipeg Free Press*, *The Star Phoenix*), on the radio (e.g., CBC Saskatchewan's *Morning Edition*, *Blue Sky*, and *The Afternoon Edition*), and on television (e.g., *CBC News: Saskatchewan* and others). With all the prairie press, new math not only became a hot topic of discussion from coast to coast to coast (e.g., articles started appearing in *The Province* in British Columbia, *The Globe and Mail* in Ontario, and elsewhere) but it concurrently introduced the country to the Western Initiative for Strengthening Education in Math—better known as WISE Math—who, without opposition, quickly became the *de facto* voice for the back-to-basics movement in mathematics teaching and learning in Canada. With the sheer amount of press being garnered by WISE Math, the Saskatchewan Government soon began to have second thoughts about the new mathematics curriculum they had just finished introducing.

In December 2011, three months after the publication of the FCPP report, the Saskatchewan Government ordered a review of mathematics instruction and curriculum. Russ Marchuk and (Saskatchewan Roughrider legend) Gene Makowsky, then Saskatchewan MLAs, began the process of consulting students, parents, teachers, and administrators. In May of 2012, less than half a year later, Saskatchewan's Ministry of Education declared that the new math curriculum would not change. (As for the report, don't even bother asking for it. I have repeatedly emailed asking for the report, to no avail. The next step, I guess, would be to file a Freedom of Information request, but based on rumours that the report doesn't actually exist, it may not be worth one's time.) Despite the conclusion that Saskatchewan's mathematics curriculum was in line with what others were doing across North America, WISE Math was further thrust into the media spotlight.

Two of the WISE Math founders, Anna Stokke (University of Winnipeg) and Robert Craigen (University of Manitoba), quickly became media darlings. For example, Professor Stokke appeared in *Maclean's*, on CBC Radio's *Cross Country Checkup* (then hosted by Rex Murphy), *The National*, *Lang & O'Leary Exchange* (Kevin O'Leary wasn't there on the day Anna appeared), and *Definitely Not The Opera*. She also penned articles for several Canadian newspapers, including the *National Post* and the *Winnipeg Free Press*. Next thing you know, circa 2012, the teaching and learning of mathematics became a national news story. (Awards for her efforts would quickly follow.) Soon, other provinces entered the mix.

In 2012 and 2013, certain Atlantic provinces that had adopted the WNCP common curricular framework, such as Newfoundland and Nova Scotia, also came under fire from WISE Math. Sure enough, the back to basics critics moved their ire to these provinces.

Running in *The Chronicle Herald*, Mr. Zwaagstra penned an article entitled "How to make math education worse in N.S." Concurrently, in what you could call a win for WISE Math, Manitoba decided to make changes to their existing mathematics curriculum, which the media (e.g., *Winnipeg Free Press*) would quickly story as the province ditching new math for back-to-basics. Worthy of note, there is no mention of WISE Math or Anna Stokke et al. in the Acknowledgements, Preface, or References in the revised curriculum (not anywhere?!). The story of Manitoba going back to basics would be picked up again by Wente and Alphonso and Stokke at *The Globe* and Woods for *National Post*. Looking back, things had been, believe it or not, relatively calm up to this point, but all that would change when the 2012 PISA results were released. Enter Ontario and Alberta into the mix.

Every four years, the Organisation for Economic Co-operation and Development assesses 15-year-old students worldwide in reading, mathematics, and science via the Programme for International Student Assessment (PISA). In the 2012 assessment, which had a specific focus on mathematics, Canada dropped slightly in the standings but remained above average. To hear the media report on the issue, however, was quite a different story. Two key points were hammered home by the usual suspects: First, depending on who you asked, we were purportedly in the midst of either a national emergency or a national disaster. Second, discovery math/learning (the now-accepted antonym for back-to-basics) was squarely to blame. Clearly, the PISA results fanned the flames of The Canadian Math Wars. The reporting at this time got a tad hyperbolic.

> "Depending on who you asked, we were purportedly in the midst of either a national emergency or a national disaster. And discovery math was squarely to blame."

According to a former Deputy Prime Minister, John Manley, the PISA results should have been considered a national emergency. I should point out here that Canada does have a National Emergency Response System. Having pored through the document, I contend that a slight dip in standardized test scores is not, in fact, a National Emergency. Similarly, other outlets reported the PISA results as a national disaster. You guessed it, I also looked into Canada's National Disaster Mitigation Program. Once again, the issue of math test scores of 15-year-old Canadians was nowhere to be found in the document. Perhaps the most hyperbolic of all the statements also came from John Manley: "How can we be satisfied with 13th place in math when we're not satisfied with second place in hockey?" How?! If only Canadians were as passionate about mathematics education as they are about the game of hockey—if only. Statements such as these were tempered best, in my opinion, by the words of Professor David Wagner in a short essay submitted, but not published in *The Globe and Mail*:

> Metaphorically speaking, we may be at the top of the class, but it is a class full of idiots. Let me explain. The PISA results make Canada look good internationally. Ironically, our nation's response last week exposes us for being poor mathematical citizens. We do not know how to read statistics and we are not equipped to think critically about "expert" reports on statistics. The PISA results position Canada at the top for sustained mathematical thinking (which is related to critical thinking), yet our response to the PISA report shows that we have a long way to go. We do not understand statistics and we do not know how to question self-proclaimed experts.

It should be noted that much of the clout originally associated with WISE Math—clout subsequently cemented by the reporting by the media of the PISA results—came from the online petition portion of their website. As is the case, many signatures indicating support for an initiative open many doors. And WISE Math had signatures. In the age of social media, though, petitions are no longer the purview of organizations or groups or associations. Today, any individual can start a petition. Case in point, circa 2014, Nhung Tran-Davies, MD, began a back-to-basics petition in Alberta and retired school teacher Teresa Murray began her, similarly themed, petition in Ontario (and, later, Tara Houle in British Columbia). In Alberta, Dr. Tran-Davies had amassed enough signatures that, metaphorically speaking, the office door of then Education Minister Jeff Johnson swung open.

As The Canadian Math Wars marked their territory in Alberta and Ontario, conversations on the topic began to shift beyond mathematics. As a person born in British Columbia, I originally blamed the political climates found in Alberta (the frozen Texas of Canada) and Ontario (the landmass surrounding the province of Toronto) for the difference. Case in point: David Staples, working for the *Edmonton Journal*, ran a 44-part series in the online version of the newspaper that he called *The Great Canadian Math Debate*. A quick glance of the headlines shows that approximately half of the articles in the series only used the debate about the teaching and learning of mathematics as a gateway to discussions about political parties in Alberta, Alberta Education, and the individuals involved in the former and latter. Perhaps the most telling headline: "Fix Alberta's math curriculum or step aside, math basics advocates to Education Minister Jeff Johnson." As to whether the series was biased in one way or the other, I cherry-picked this headline: '"This new math is stealing their confidence and their dreams'—educators speaks [*sic*] out against new fuzzy math curriculum." (And this one: "Memorizing times tables is damaging to your child's mind, discovery math expert says.") In Alberta, in 2014, The Canadian Math Wars began to be subsumed by the Canadian Culture Wars. To give an illustrative example of this shift: Remember the rally?

> "In Alberta, in 2014, The Canadian Math Wars began to be subsumed by the Canadian Culture Wars. To give an example of this shift: Remember the rally?"

I would be remiss not to mention that in 2014, a rally was held in front of the Alberta legislature. No, the rally was not held to support (or oppose) some oil-related cause. And no, the rally was not held in order to preserve the Alberta provincial tax rate of 0%. No—that cool, damp morning in April of 2014, over 100 individuals, with signs and loudspeakers and the whole shebang, rallied to oppose the so-called new math. And, with that rally, any doubt that the debate over teaching and learning mathematics had become a politicized issue was all but removed, in my opinion.

In 2014, the year the back-to-basics rally was held at the Alberta legislature—and the year that article after article after article supporting the back-to-basics movement from the likes of Zwaagstra, Craigen, Staples, Tran-Davies, Alphonso, Houle, Wente, Murray, and Stokke appeared in major newspapers—the annual meeting of the Canadian Mathematics Education Study Group (CMESG) was held in Edmonton. CMESG, a group of mathematics educators, mathematicians, and mathematics teachers that meet annually, held a panel on the debate over the teaching and learning of mathematics. The conclusion of the panel: while PISA scores, yes, had fallen, there was no crisis; and, despite what was being presented in the media, discovery math was being erroneously blamed for the non-existent crisis. As for how this message was received by the general public in Alberta, the comments sections of the *Edmonton Journal* and *Edmonton Sun* confirmed, at least for me, why certain publications like *Popular Science* have turned off their comments section. Here is my favourite comment on the CMESG story from ndtguy:

> Oh please. New math is ridiculous. 'The Expert is a idiot. Discovery learning is a joke for all the touch feely people out there who are afraid to hurt someone's feelings by saying they are wrong. Disovery learning does nothing to prepare any student for rough life in post secondary or the REAL world. The minute anyonye tells them the are wrong. I'm sure their whole world will come crumbling down. Good luck to us all.

Good luck to us all, indeed.

The same year that saw Ontario, Quebec, and British Columbia join Saskatchewan in keeping their current mathematics curricula, Alberta, like Manitoba, made some changes to theirs. Alberta decided to change the language in their curriculum to support memorization of times tables (which was the hot-button math issue at the time), but with the caveat that newer methods of teaching, those at the core of the debate, would not be eliminated. Similar changes were taking place in Ontario.

Believe it or not, the province of Ontario looked to the province of Manitoba for guidance when it came to the debate over the teaching and learning of mathematics. Kathleen Wynne, then Premier of Ontario, was a former public-school trustee in Toronto's Ward 8, parliamentary assistant to Minister of Education Gerard Kennedy, and, in 2006, Minister of Education, which meant that she was considered uniquely qualified to rectify the mathematics kerfuffle that was taking place in the province. Despite dedicating 4 million dollars to address the issue, having some 3000 math teachers attend summer school to improve their skills, and launching other initiatives, the yearly math scores from the province's Education Quality and Accountability Office and associated comments in the media did not help with Wynne's efforts at reelection. In 2018, Wynne suffered the worst defeat of a governing party in Ontario history to, you guessed it, Douglas Robert Ford, which brings us back to my contention regarding the Canadian Math Wars being subsumed by the Canadian Culture Wars.

> "Another difference in the last few years is a new voice beginning to appear in the media—the voice of mathematics educators."

To be honest, the conversation regarding the Canadian Math Wars from 2015 to 2018 was nowhere near as fast and furious as it was from 2011 to 2014. Sure, the online petitions are still up in Manitoba, British Columbia, Alberta, and Ontario. Dr. Tran-Davies' petition was brought to the attention of Alberta's new Education Minister, Jim Prentice. And of course, the usual suspects continue to periodically let the country know that the sky is still falling. This reporting, though, now mostly takes place at the beginning of the school year. Stories that would have once made national headlines (for example, British Columbia's recent introduction of a new mathematics curriculum) now receive little attention, relatively speaking. Oh yeah, CBC's *Cross Country Check Up* reran their math show from 2011. Another difference in the last few years is a new voice beginning to appear in the media—the voice of mathematics educators.

In what I deem WISE Math's greatest mistake, Anna Stokke wrote a report for the C. D. Howe Institute, where she opined for an 80/20 split between traditional and discovery techniques (80% for the former, 20% for the latter). Unlike in years past, when the word of WISE Math was treated as sacrosanct by the media, Stokke and other members of WISE Math were taken to task in both traditional and social media. Sure, individuals like Staples and others were in support of Stokke's recommendation, but this is not a surprise. On the other side, other university professors, those who had been silent for years as the one side of the debate dominated the media, finally began to speak out when the findings of Stokke's report were released. Their main issue: Stokke's proposed 80/20 split showed up out of thin air, which all considered hypocritical given the back-to-basics hardened stance that educational research isn't 'real' research. Too little, too late for the newcomers to the debate? Sure, but as many of them told me personally, they were aware of the topic but weren't interested in rolling around in the muck (see, for example, the comment on the CMESG findings). Speaking of rolling around in the muck, via social media, Robert Craigen and well-known American mathematics education Messiah, Dan Meyer, debated Stokke's 80/20 split over Twitter for a period of nearly three straight days.[1] Other than heated debates by individuals heavily vested in the topic, the Canadian Math Wars have been relegated to the political arena.

> "Other than heated debates by individuals heavily vested in the topic, the Canadian Math Wars have been relegated to the political arena."

Evidence of this is found throughout the media. Headlines refer to Education Ministers of varying political parties toeing the company line. Whether Conservative, Liberal, or NDP, whether in British Columbia, Alberta, Saskatchewan, Manitoba, Ontario, or the Atlantic Provinces, politicians are well aware of the debate over the teaching and learning of mathematics and have varying ways of addressing the issue. On the one hand, and painting with a very broad brush, Liberals (in both senses of the word) are looking to address the current situation by tweaking what already exists. For example, the previous Liberal Government in Ontario required each school to have a math leader teacher, dedicated $60 million to improve mathematics teaching and learning, and made other efforts such as mandatory numeracy and mathematics tests for future teachers. More generally speaking, they are looking to build upon what exists while, at the same time, attempting to fix issues that have arisen. On the other hand, the Conservative playbook is simpler. Step 1: Tear it all down. Step 2: Get back to basics. That brings me back to Douglas Robert Ford.

As I said at the beginning, I've been following this story closely for the last seven years. In that time, through the use of technology (RSS Readers, etc.), I've been able to read nearly every story that has come out on the debate in our country. In addition to reading all these news stories, I've flown to British Columbia and Ontario to give keynotes and presentations on this topic for various organizations and associations, and written articles in journals and chapters in books. To stay best informed on the topic, I'm also keeping tabs on this debate in other countries like the USA, UK and Australia. Of all the material that I've encountered, a comment on an article[2] published in *The New York Times*, of all the things that I've read,

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[1] A record of this conversation was available at www.storify.com/matthewmaddux/rcraigenvsddmeyer, before the Storify platform was shut down.

[2] Crary, A., & Wilson, W. S. (2013, June 16). The faulty logic of the 'math wars.' *The New York Times*. Retrieved from https://opinionator.blogs.nytimes.com/2013/06/16/the-faulty-logic-of-the-math-wars/

has stuck with me the most. According to a Joe from Raleigh, NC:

> I grew up in the 1950's and know little of how math is taught now, but this debate seems sickeningly familiar: In the late 60's and the '70's we had New Math, a trend away from rote memorization, toward learning the "whys." The pedagogical debate became part of the Culture Wars: If I knew your position on Vietnam or Black Power, I knew your position on the New Math – just like today!

Like Joe, if I know your positions on, say, domestic policy, social issues, immigration issues, foreign policy, healthcare, the economy, the environment, electoral issues, and others, I'm fairly confident I know your position on the new math. After all, the Canadian Math Wars are now just another cog in our ongoing Canadian Culture Wars. It will be interesting, at least for me, to see if the Canadian Math Wars plays a part in party platforms, promises, and policies during the impending 43rd Canadian federal election.

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Ilona & Nat,
Editors

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