The Variable

Presented by the Saskatchewan Mathematics Teachers' Society

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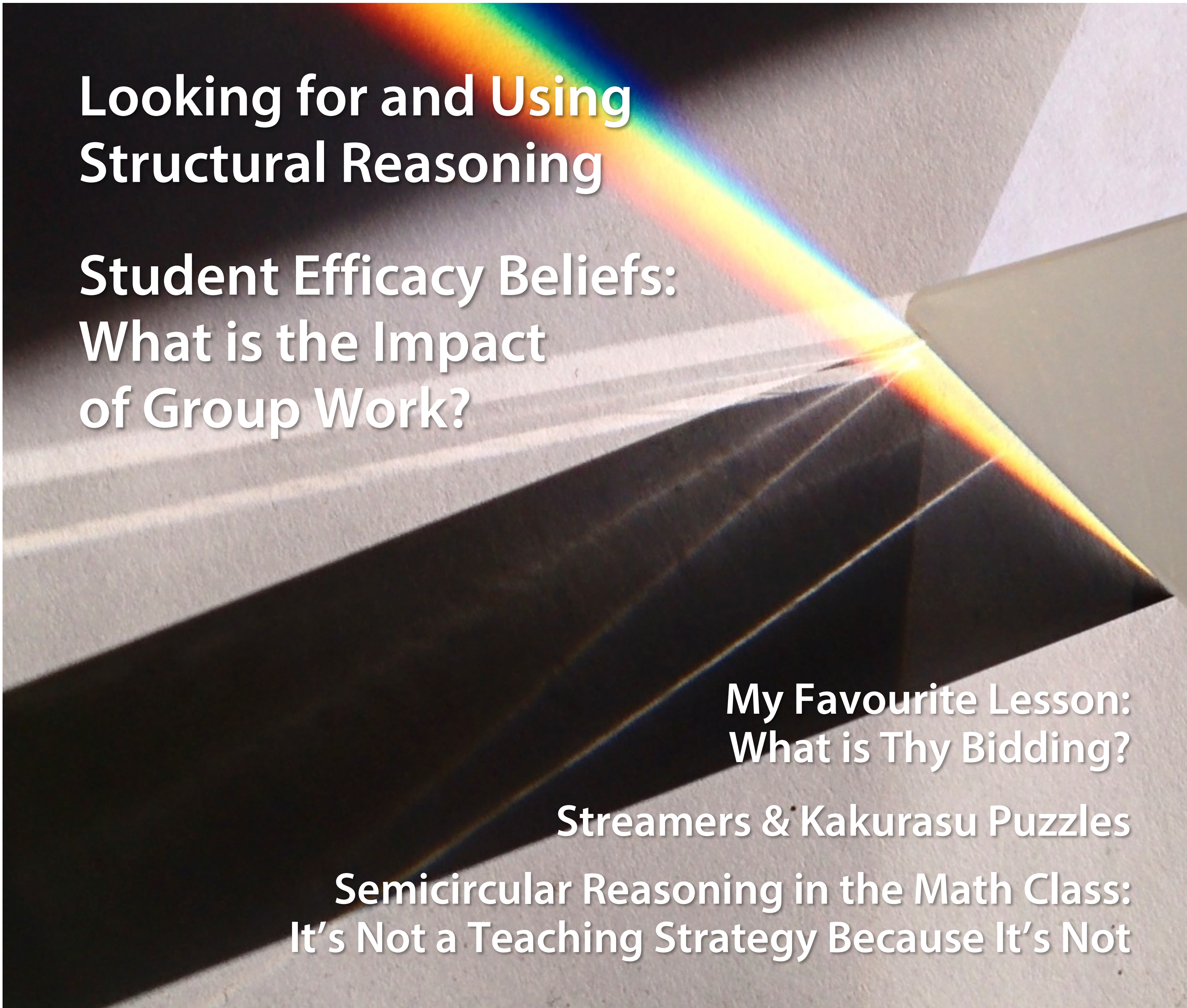

Looking for and Using Structural Reasoning

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Semicircular Reasoning in the Math Class: It's Not a Teaching Strategy Because It's Not

Saskatchewan Mathematics Teachers' Society

AFFILIATE
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Notice to Contributors

The Variable welcomes a variety of submissions for consideration from all members of the mathematics education community, including classroom teachers, consultants, teacher educators, researchers, and students of all ages. Submitted material is assessed for interest, presentation, and suitability for the target audience.

Please submit articles by email to thevariable@smts.ca in Microsoft Word format. Authors should aim to limit submissions to 3000 words or less and include a photo and a short biographical statement of 75 words or less with their submission. Accepted contributions are subject to revision. Editors reserve the right to edit manuscripts for clarity, brevity, grammar, and style.

Editorial

As the calendars turn to the year 2020 and "perfect vision" puns begin to saturate advertising agencies' attempts to grab our attention, we too feel it insightful[1] to embrace the vision-based puns offered by the calendar to situate the work of *The Variable* alongside the daily lives of Saskatchewan teachers. Education, in a general sense, has been thrust into the public eye[2] with increasing frequency in recent months. In fact, with the pace that news circulates in today's interconnected world, it is nearly impossible to remain blind[3] to the very public debates in some Canadian provinces about the state of education—although most of the press seems more concerned with political optics[4] and pocketbooks than with any real concern for teaching and learning.

Improvement is often pursued through wide-angle initiatives, as demonstrated by the provincial committee convened just this month to develop recommendations on curriculum development and high school graduation requirements in Saskatchewan .These simultaneously feel important, yet seem to put the focus[5] on learning as legislated, leaving the lived experiences of teachers and learners in the periphery[6].

This leaves teachers in a balancing act, with the wide lens[7] of policy and politics on one side of a scale counter-weighted by the daily concerns that come with being charged with the education of a specific group of learners with specific histories and specific needs. *The Variable* aims to amplify[8] and support the daily work of teachers, the stakeholders who are uniquely situated with a view[9] of the nexus between legislation and lived experience. Ultimately, it is the perspective[10] of the teacher that provides a point of clarity[11] we cannot afford to overlook[12].

Nat & Ilona, Co-Editors

[1] Pun intended.
[2] Pun intended.
[3] Pun intended.
[4] Pun intended.
[5] Pun intended.
[6] Pun intended.
[7] Pun intended.
[8] Token ear-related pun.
[9] Pun intended.
[10] Pun intended.
[11] Pun intended.
[12] Pun intended.

What is Thy Bidding?

Jared Hamilton

You and your friends are starting a landscaping company. You speak with Albert, a recently retired landscaper, for things you'll need to consider when starting your new business. Albert has been in business for over 20 years, and provides you with some information on his last two jobs (see Figures 1 and 2 and computations on p. 10-11):

Job #	Surface Area	Labor Crew	Job Time	Bid	Actual Cost
1	674 m^2	4 people	6 hours	$13 450.00	$4 482.00
2	212.75 m^2	3 people	4 hours	$5 300.00	$1 766.00

Table 1: Albert's last two jobs

His local supplier sells sod for $250.00 per pallet. One pallet = 48 square metres and takes one person approximately 1.5 hours to install. He pays his employees $18.00 per hour and charges each job a fixed operation cost of $300.00. Albert advises the group to do their research and build an appropriate business model that accounts for their costs of operation and desired profits while being competitive.

The Task

This activity allows students to create and design a business model for a landscaping company. Using the information provided above, students will:

- be randomly organized into teams of 2-3 and create a business name;
- develop a model to provide a quote for future potential jobs that needs to account for appropriate costs such as labour (wages, overtime, etc.) and materials (sod, fertilizer, equipment, etc.);
- design the quote to provide a margin of profit (students may organize their work on the quote sheet provided on p. 9);

- use their model to bid for two upcoming jobs: the first includes measurements, and the second is a scaled diagram that needs to be measured (see p. 7 and 8); and
- provide the bid to the customer in format of a business quote that the surface area of job, the approximate time to complete, and the approximate cost.

Curricular Competencies and Content

Curricular competencies include:

- collaboration and problem solving;
- reasoning and modelling;
- understanding and solving;
- communicating and representing;
- connecting and reflecting;
- incorporating elements of technology.

Content includes:

- measurement and design (WA 10.4[1], WA 10.5[2]);
- rates and proportion (WA 10.10[3]);
- financial literacy (WA 10.11[4]);
- numeracy and operations;
- estimation and rounding.

Assessment

Emerging/ Developing	Criteria/Competency	Proficient
	Communicate ideas and mathematical thinking involved in project design	
	Engage in mathematical thinking in choosing what content to discuss and explore based upon the design of the project	
	Model mathematics in contextualized experiences such as ratio, measurement, finance, numeracy	

[1] WA10.4: Demonstrate, using concrete and pictorial models, and symbolic representations, understanding of linear measurement, including units in the SI and Imperial systems of measurement. (Ministry of Education, 2010, p. 29)

[2] WA10.5: Demonstrate using concrete and pictorial models, and symbolic representations, understanding of area of 2-D shapes and surface area of 3-D objects including units in SI and Imperial systems of measurement. (Ministry of Education, 2010, p. 31)

[3] WA 10.10: Apply proportional reasoning to solve problems involving unit pricing and currency exchange. (Ministry of Education, 2010, p. 35)

[4] WA10.11: Demonstrate understanding of income including: wages; salary; contracts; self-employment; gross pay; net pay. (Ministry of Education, 2010, p. 36)

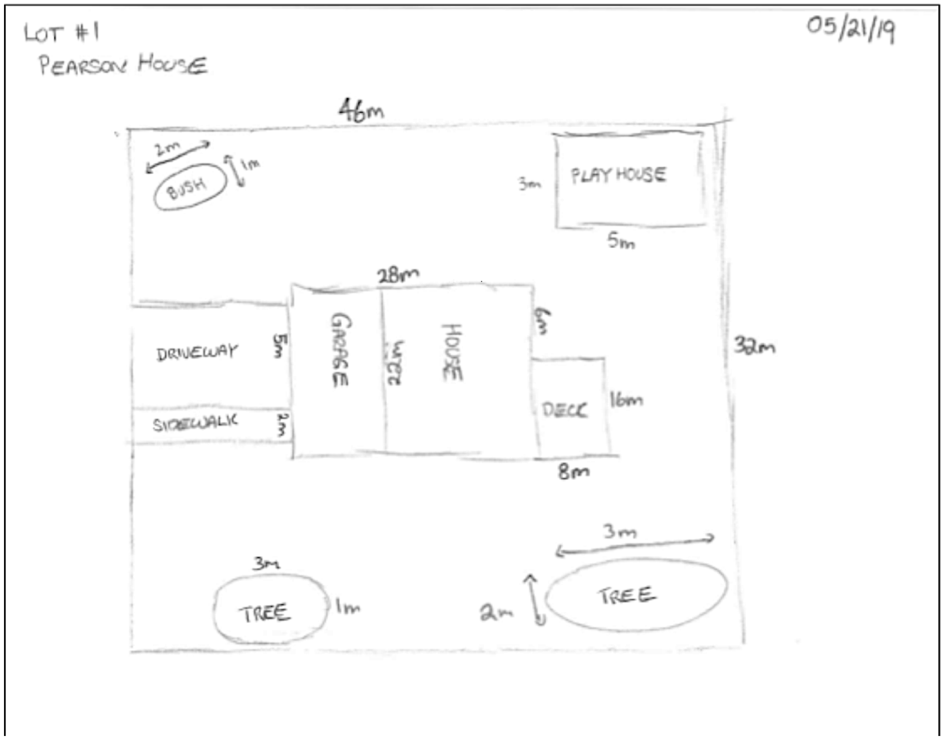


Figure 1: Job # 1

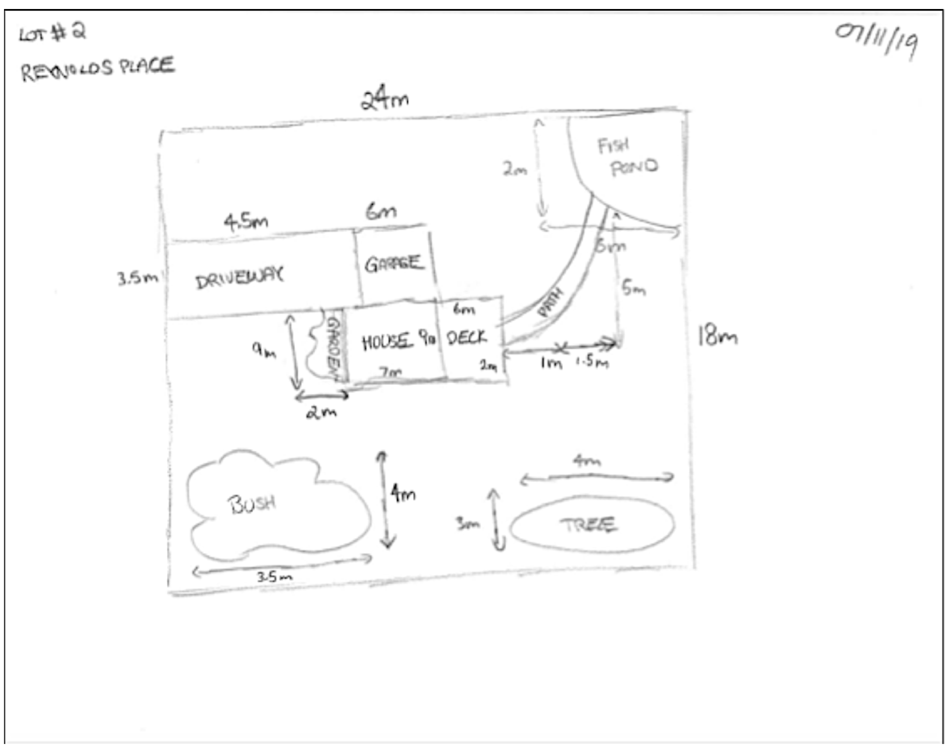
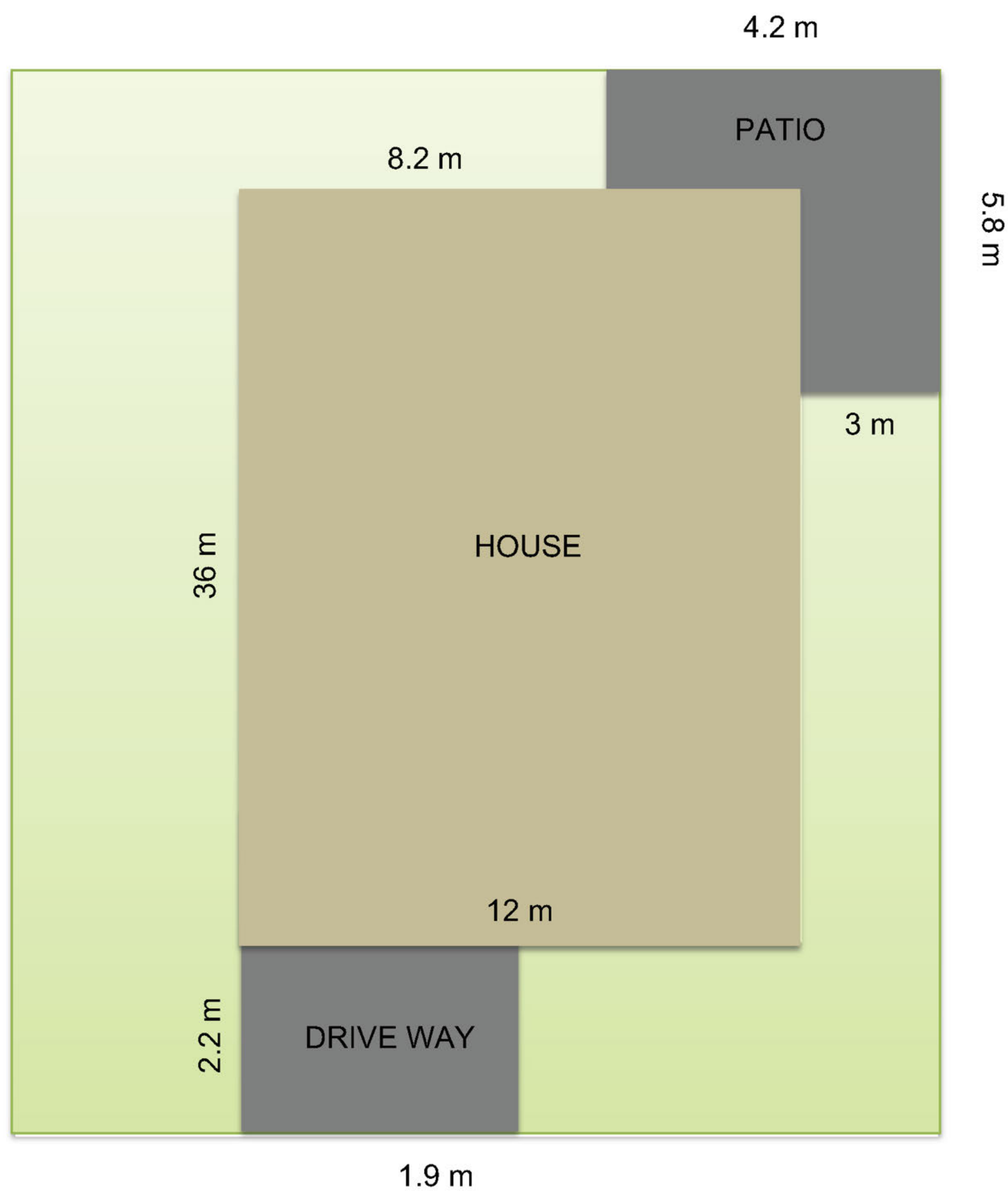
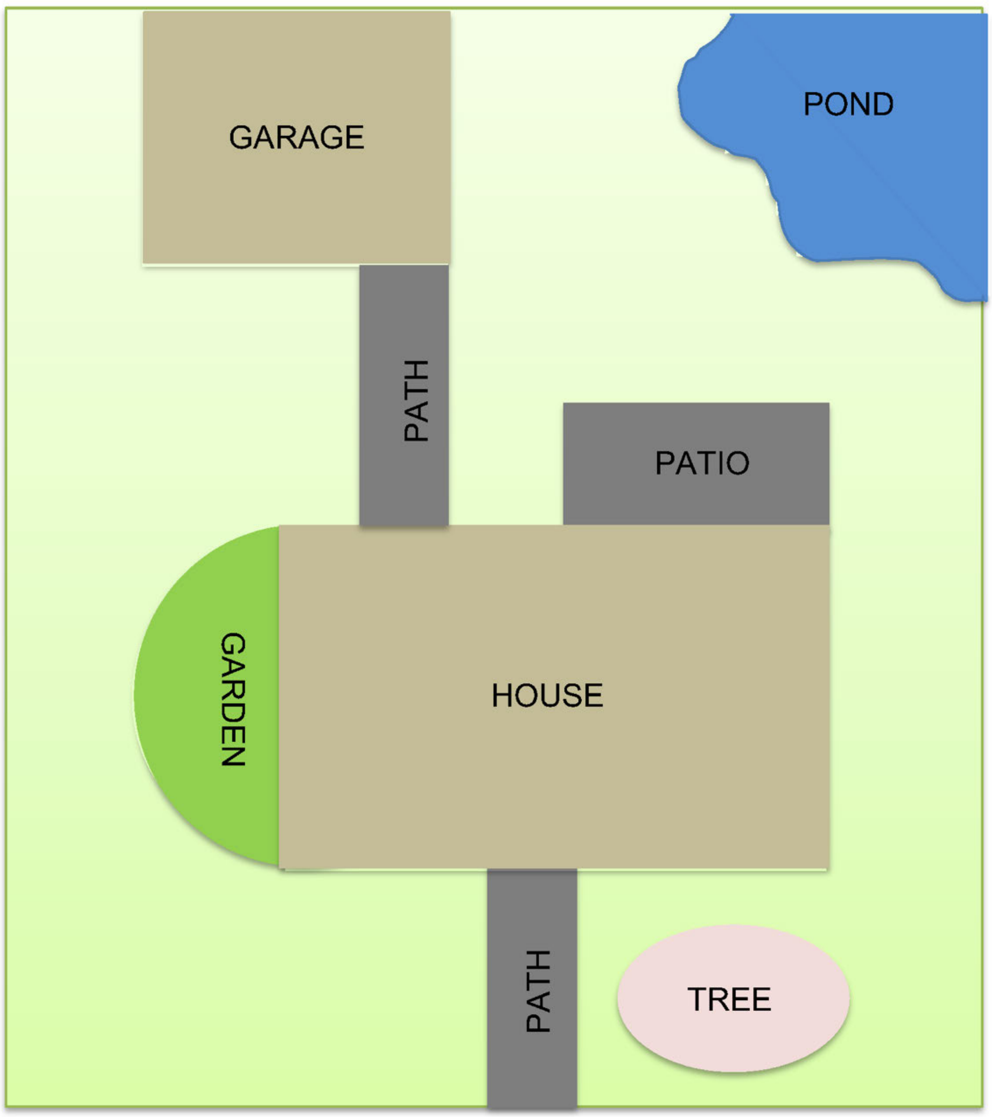


Figure 2: Job # 2

House Lot #1:
Lot dimensions: 20 m x 40 m



House Lot # 2:
Blueprint drawing
Scale 1 cm : 3.5 m



Quote Sheet for Lot # ________

Company Name: ______________________________ **Date:** ________________________

Contractors: ____________________ ____________________ ____________________

Area of worksite: ____________________

Rationale:

Estimated time: ____________________

Rationale (based on previous jobs):

Estimated cost: ____________________

Rationale (based on previous jobs):

Albert's Bids - Rationale

Albert's bids were developed according to the following rationale, with calculations included below:

- crews consist of two to four people (otherwise, a second truck is needed, which adds an extra $100.00 to the bid);
- 1-1.5 hours per person are added on a job with many irregular cuts;
- a fixed operation cost of $300.00 is added to cover expenses such as fuel, broken equipment, etc.;
- irregular shapes are squared off for quick and easy calculation (the faster the quote is sent to the customer, the more likely the company will win the bid);
- a 5% error is assumed for the sod size;
- the bid is calculated as cost × 3 (this covers cost, taxes, and the business owner's salary).

It is intended that the teacher use this information as necessary to guide students' discussion about developing a bid and to help them identify costs they may not have considered, but also that the teacher allows students to develop their own models using the information provided by Albert in Table 1 and in Figures 1 and 2.

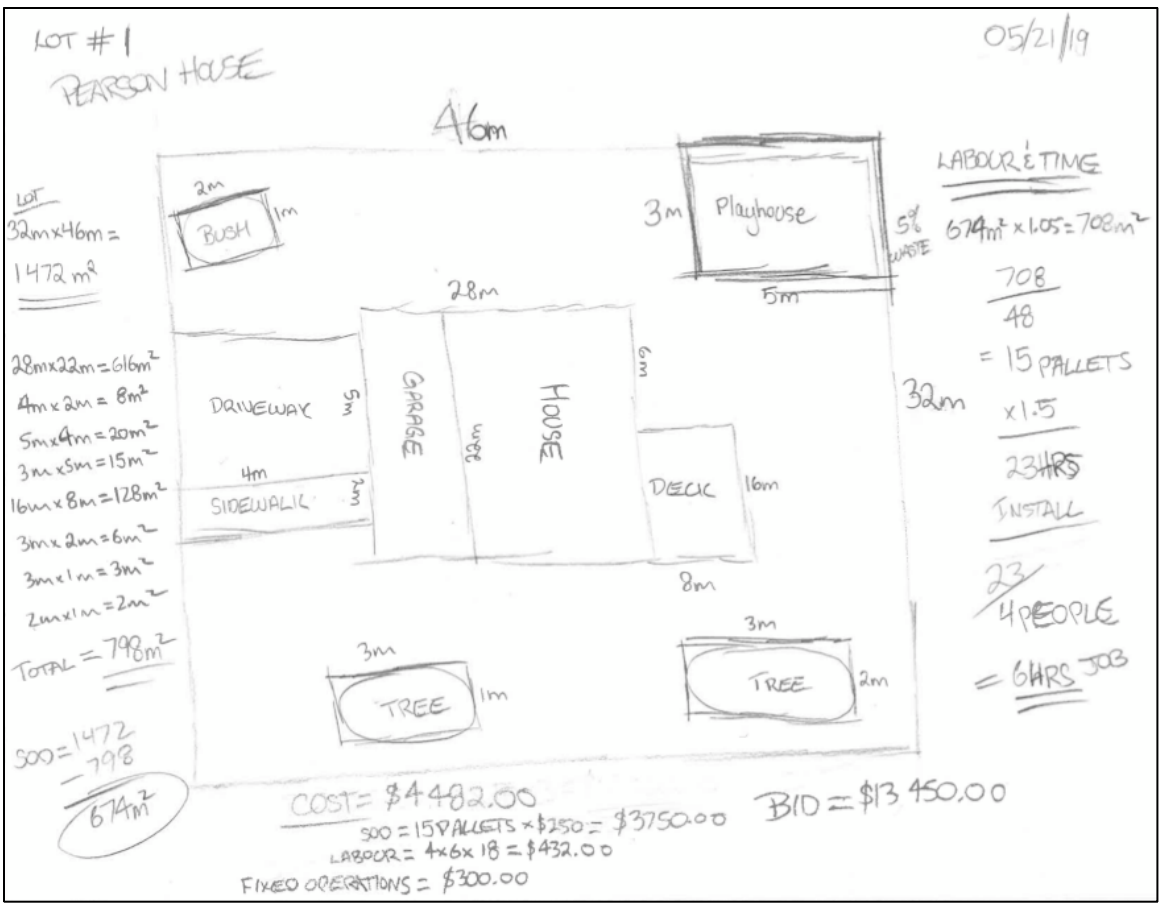


Figure 3: Calculations for Lot # 1

Figure 4: Calculations for Lot # 2

Jared Hamilton is a numeracy & technology teacher in Northern British Columbia. Over the past decade he has taught in both Canada and the United Kingdom, and is starting his phD in education. Interests include his family, learning with his students, the Edmonton Oilers, coffee and donuts.

Contribute to this column!

The Variable exists to amplify the work of Saskatchewan teachers and to facilitate the exchange of ideas in our community of educators. We invite you to share a favorite lesson that you have created or adapted for your students that other teachers might adapt for their own classroom. In addition to the lesson or task description, we suggest including the following:

- Curriculum connections
- Student action (strategies, misconceptions, examples of student work, etc.)
- Wrap-up, next steps

To submit your favourite lesson, please contact us at thevariable@smts.ca. We look forward to hearing from you!

Alternate Angles is a column on problems from multiple perspectives: various methods that could be used to solve them, insights we get from their solution, the new paths that they can lead us to once they have been solved, and how they can be used in the classroom.

A Slippery Slope

Shawn Godin

Welcome back and Happy New Year, problem solvers! Last issue, I left you with the following problem:

> The following information is known about ΔOBC:
>
> - O is at the origin, and points B and C lie in the first quadrant;
> - ΔOBC is an isosceles right triangle with $OB = BC$ and $\angle OBC = 90°$; and
> - the hypotenuse OC is on a line segment with slope 3.
>
> Determine the slope of line segment OB.

This problem was a Problem of the Week from the Centre for Education in Mathematics and Computing (CEMC) at the University of Waterloo. The CEMC produces competitions as well as resources for teachers and students in the areas of mathematics and computer science. The Problem of the Week is one of those resources. It is released weekly, with a solution (and new problem) provided the following week. The problems come in five groups based on the grades of the targeted students: Group A – Grades 3 and 4; Group B – Grades 5 and 6; Group C – Grades 7 and 8; Group D – Grades 9 and 10; and Group E – Grades 11 and 12. In many cases, the problems for many of the groups have similar contexts, so you have access to easier and more challenging versions of the problem you are considering. The problem above was a problem from the 2018-2019 school year for Group E. You can check out the Problems of the Week and other CEMC resources at www.cemc.uwaterloo.ca.

One way to start this problem is to physically draw the triangle in question. In Figure 1, I have drawn a line of slope 3 that passes through the origin on graph paper. Then, using a 45º- 45º - 90º triangular ruler found in a standard "math set," I drew in our triangle. By searching for points on OB, it looks like (4, 2) is on the line, so the slope of OB is $\frac{1}{2}$.

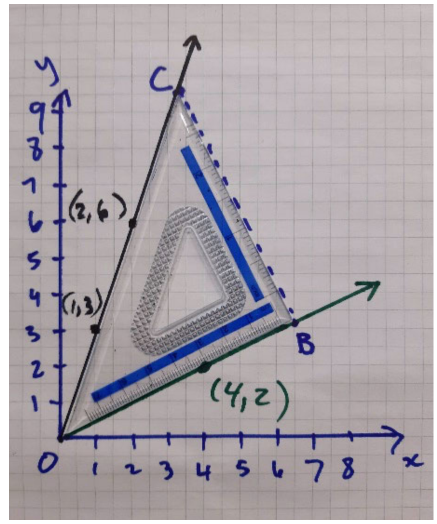


Figure 1: Drawing a solution

In the first solution, the accuracy of my answer depends on the accuracy of my diagram. What this allows me to say is that the slope is probably *around* $\frac{1}{2}$, but my result may not be exact. Could I tell the difference between a slope of 0.47 and a slope of $\frac{1}{2}$ on my diagram? Probably not. So, lets dig a little deeper.

We will use some geometric properties to aid our solution. A useful property of isosceles triangles is the fact that the angle bisector of the apex (the "non-equal" angle, in a non-equilateral isosceles triangle, as in our case) coincides with the median from the apex, the altitude from the apex, and the perpendicular bisector of the base (the "non-equal" side). Using this, we can use dynamic geometry software to model our situation. Figure 2 is a screenshot of a sketch I created in Geogebra. In the sketch, I have constructed the segment OC, with C at (2, 6), which will serve as the hypotenuse and the base of our isosceles triangle. Thus, the perpendicular bisector will be the line through M(1, 3) with slope $-\frac{1}{3}$, which has equation $y = \frac{10}{3} - \frac{x}{3}$ and is indicated by the red dashed line. We can then construct a point B on the perpendicular bisector and the segments OB and BC, to complete our triangle. We can then measure the angle $\angle CBO$ and the slope of OB, which are 84.86° and 0.45, respectively, in the diagram. Point B can then be moved along the perpendicular bisector until $\angle CBO = 90°$, at which point we can see that the slope is $\frac{1}{2}$, as we suspected earlier.

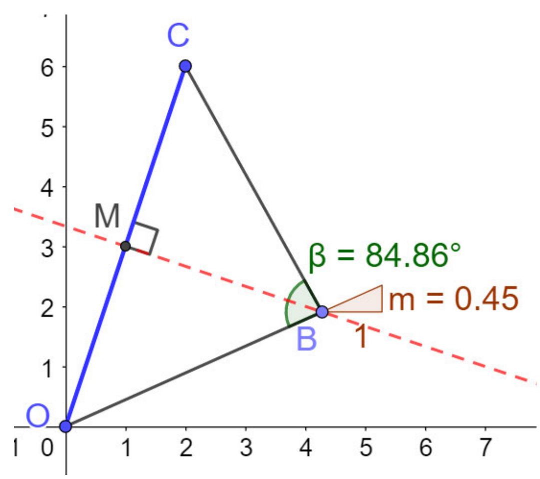


Figure 2: Drawing a solution with Geogebra

We can use another geometric property to our advantage. If we draw a triangle so that its three vertices are on a circle, then if one of the sides is a diameter of the circle, the opposite angle measures 90°. This property is usually stated as "the angle inscribed in a semicircle is a right angle." When I was in school, we studied more geometry than is currently taught. You can see the proof of the theorem and a few more circle properties in a recent article of mine (Godin, 2019). Figure 3 shows that if we construct the circle with OC as a diameter, then the point of intersection of the circle with the perpendicular bisector of OC is our point $B(4,2)$, from which we can again verify that our slope is correct.

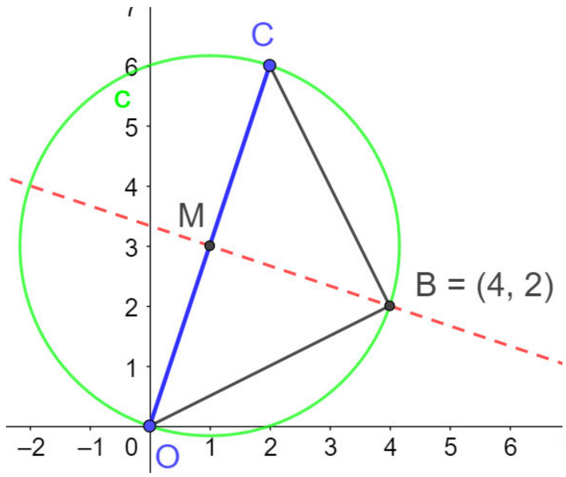


Figure 3: Using properties of circles

Using the properties of circles allows us to determine the point algebraically as well. We already know the equation of the perpendicular bisector. The circle has centre $M(1,3)$ and radius OM with length $\sqrt{10}$. Hence, the equation of the circle will be $(x-1)^2+(y-3)^2=10$. Substituting in the equation of the perpendicular bisector yields

$$(x-1)^2+\left(-\frac{1}{3}x+\frac{1}{3}\right)^2=10,$$

which we can solve by expanding, gathering like terms, and so on. However, observing that $-\frac{1}{3}x+\frac{1}{3}=-\frac{1}{3}(x-1)$, we can simplify the equation as follows:

$$(x-1)^2+\frac{1}{9}(x-1)^2=10$$
$$\frac{10}{9}(x-1)^2=10,$$

which leads to $x-1=\pm 3$, yielding *two* solutions: $x=4$ (leading to the solution we have been exploring) and $x=-2$, which is not in the first quadrant, as the problem stipulates.

Looking back at Figure 3, we can indeed see that there is a point that would create another right isosceles triangle, albeit in the wrong quadrant. This point, $B'(-2,4)$, is the vertex of the square $OBCB'$ inscribed in the circle with diameter *OC*. As such, OB' has slope -2 as it is perpendicular to *OB*. On the other hand, $B'C$ is parallel to *OB* and also has slope $\frac{1}{2}$.

We can also solve the problem using some trigonometry and trigonometric identities. If we let θ be the angle between *OB* and the positive *x*-axis, then the slope of *OB* is $\tan\theta$. Since the triangle is right isosceles, $\angle BOC=45°$. So, since the slope of *OC* is 3 and the angle *OC* makes with the positive *x*-axis is $\theta+45°$, we must have $\tan(\theta+45°)=3$. Using the identity for the tangent of a sum, we get

$$\frac{\tan\theta+\tan 45°}{1-\tan\theta\tan 45°}=3.$$

Since $\tan 45°=1$, this simplifies to

$$\frac{\tan\theta+1}{1-\tan\theta}=3,$$

from which we can extract that $\tan\theta=\frac{1}{2}$, our desired slope.

I will sketch one more possible solution. This was the first of three solutions presented by the CEMC (the other two are similar to two of my solutions). Let B have coordinates (p, q) and inscribe triangle OBC in a rectangle whose sides are parallel to the coordinate axes. Then the two shaded triangles in Figure 4 have a specific relationship that allows us to determine the coordinates of C in terms of p and q. Since the slope of OC is known, we can calculate a relationship between p and q which allows us to determine the slope. I will leave the details to you; you can check out the solution at the CEMC website[1].

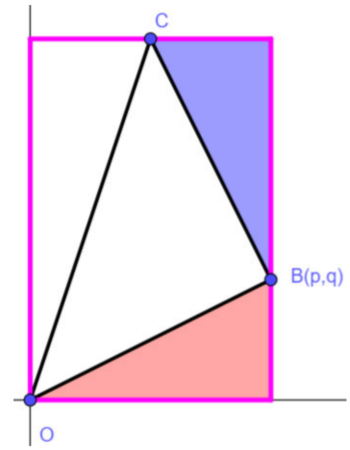


Figure 4: One last solution

Another satisfying problem that admits solutions of many different types and levels of sophistication. This allows many students to have the ability to attack the problem. On the other hand, students with a broader mathematical background can possibly generate multiple solutions and reflect on their merits.

And now, it's time for your homework:

> Each of the variables P, Q, R, S, and T is a digit in the two six-digit numbers appearing in the product below. Determine the values of these variables.
>
> $$\begin{array}{rcccccc} & P & Q & R & S & T & 4 \\ \times & & & & & & 4 \\ \hline & 4 & P & Q & R & S & T \end{array}$$
>
> Until next time, happy problem solving!

References

Godin, S. (2019). Problem solving vignettes: Going in circles. *Crux Mathematicorum*, 45(8), 452-456. Available at https://cms.math.ca/crux/v45/n8

Shawn Godin teaches at Cairine Wilson Secondary School in Orleans, Ontario. He strongly believes in the central role of problem solving in the mathematics classroom. He continues to be involved in mathematical activities: leading workshops, writing articles, working on local projects and helping create mathematics contests.

[1] https://www.cemc.uwaterloo.ca/resources/potw.php

Looking for and Using Structural Reasoning [1]

Casey Hawthorne & Bridget K. Druken

Both the Common Core State Standards (CCSSI 2010) and the NCTM Process and Content Standards distinguish between Standards for Mathematical Practice (SMP) and standards for mathematical content. We believe this distinction is important and note that students often acquire knowledge of mathematical content without necessarily developing the related mathematical practices. In fact, we would argue that students grappling with mathematical content without mathematical practices are developing a different understanding. For example, consider the following task:

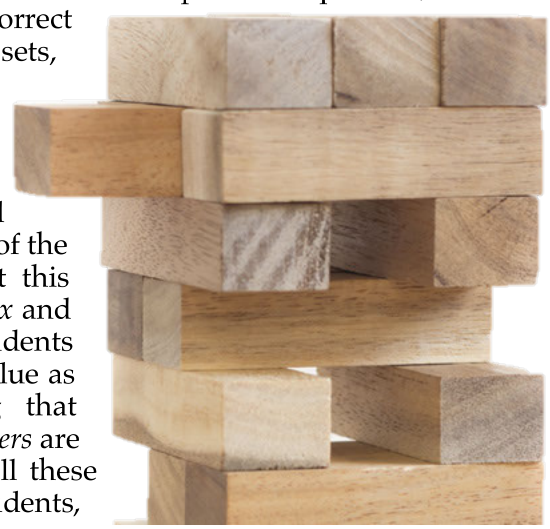
> Find a value for x that satisfies $x/(x + 3) = 1$.

Jake multiplies both sides by $x + 3$ to get $x = x + 3$, and then eliminates the x on each side. He then writes, "no solution," applying the rule he has been taught that such nonsense statements should be answered in this way.

Madi reasons that because $x + 3$ is always 3 more than x, this means that the ratio of $x + 3$ to x can never equal 1. Although both approaches arrive at a correct solution, Madi's approach invokes the mathematical practice to "look for and make use of structure" (CCSSM 2010, SMP 7, p. 8). But what does it mean to "look for and make use of structure," and how can we as teachers support students in developing this practice?

Before unpacking these questions, we offer a problem to illustrate structural reasoning. We invite you to find two ways to reason about the values of x that make the following inequality true: $|x - 3| > -4$.

This problem can be approached using algebraic techniques that leverage rules associated with symbolic notation. To do so, students might write two separate inequalities, $x - 3 > -4$ and $x - 3 < 4$, solve each separately with the correct conjunction, $x > -1$ or $x < 7$, graph their solution sets, notice that their union covers the entire number line, and conclude that the solution is all real numbers. Although this approach reflects one kind of mathematical understanding that we aspire to instill in students, this problem could also be approached by examining the structure of the inequality. Some students might observe that this absolute value will be positive for all values of x and thus will always be greater than –4. Yet other students might leverage the interpretation of absolute value as representing distance or magnitude, noting that because distance is never negative, then *all numbers* are a distance greater than –4 from 3. Although all these understandings are critically important for students,



[1] Reprinted with permission from "Looking for and Using Structural Reasoning," *Mathematics Teacher*, *112*(4), copyright 2019 by the National Council of the Teachers of Mathematics (NCTM). All rights reserved.

too often we focus on the former. These latter approaches are not only efficient but also exciting to recognize.

Structural reasoning involves first taking a step back and looking for properties that are embedded in mathematical representations before selecting a procedure to use to solve a problem. Inviting students to search for and examine relationships and properties can foster not only a greater understanding of mathematics but also a sense of self-efficacy surrounding mathematical problem solving.

Structure and Representations

In general, structure denotes characteristics of how objects are built. In mathematical terms, structure refers to the embodiment of properties and their relationships in mathematical objects. To build and access mathematics, we use representations of mathematical ideas, such as symbolic, graphical, verbal, contextual, and tabular representations. Depending on the representation used, the resulting structure takes on different forms. For instance, the symmetry of a quadratic equation can be seen more easily when graphed than in the symbolic equation $y = ax^2 + bx + c$. Because different representations offer different insights into properties (Cuoco 2001; Lesh et al. 1987), an object's structure becomes clearer when we analyze a property across multiple representations.

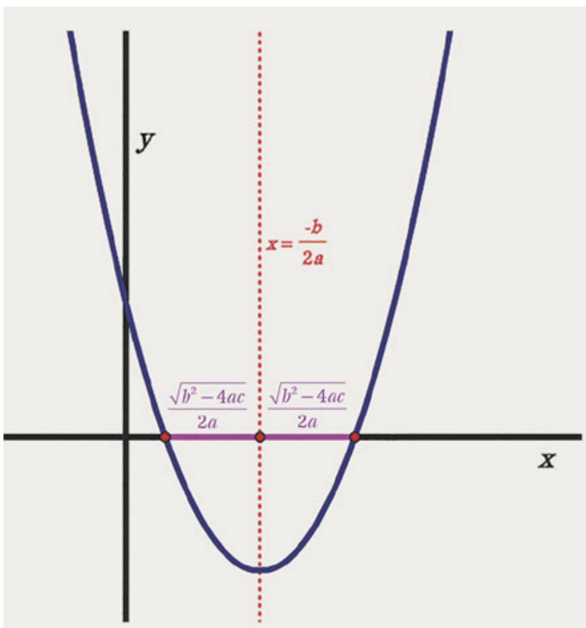


Figure 1: This representation connects a quadratic graph to the quadratic formula.

One example of how engaging with a mathematical property across multiple representations may highlight structure can be seen when finding the x-value of the vertex of a quadratic $y = ax^2 + bx + c$. Analyzing the problem *graphically*, we can see a symmetric parabola (see Figure 1). In *words*, we recognize this property by saying that the vertex lies on the axis of symmetry. Focusing on specific points, the axis of symmetry occurs halfway between the two x-intercepts. Having noticed this property graphically and described it verbally, we now look for its instantiation in *symbolic* form.

Using a lens of structure, the quadratic formula can be transformed from a rule for calculating zeros of a function to reasoning about an embodiment of symmetry. This property is highlighted by analyzing the quadratic through multiple representations. To first capture the symmetry of the parabola, we split up the quadratic formula into two fractional expressions:

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

Connecting the various parts of the expression to their graphical representations,

$$\frac{\sqrt{b^2 - 4ac}}{2a}$$

is the distance between the axis of symmetry and each of the two roots. This means that $x = -b/(2a)$ must be the center, consequently the x-value of the vertex. In summary, we can highlight structure resulting from the embodiment of symmetry within the symbolic form by describing and analyzing symmetry of parabolas through a graphical representation.

Structure and Goals

In addition to acknowledging the role that representation plays in structure, it is important to recognize that any structure that one sees in a mathematical representation will also depend on the mathematical goal. For instance, a student who does not consider the goal of a task may just see the symbolic representation of the expression $4x^2 - 9$ as a collection of disconnected symbols: 4, x, 2, and 9. It is not until the representation intersects with the goals that a student may begin to see structure in the notation.

- Goal: to factor the expression $4x^2 - 9$.

We can view $4x^2 - 9$ as the difference of two squares. This may be conveyed more clearly as $(2x)^2 - 3^2$, which can be factored as $(2x + 3)(2x - 3)$.

- Goal: to solve an equation using the quadratic formula.

We could overlay the general symbolic form of $ax^2 + bx + c = 0$ onto the given algebraic expression. Students might see that the absence of a middle term bx means a coefficient of zero and the operation of subtracting 9 as a constant term of -9. Such a view might be highlighted by rewriting $4x^2 - 9$ as $4x^2 + 0x + -9$. We could then evaluate the quadratic formula where $a = 4$, $b = 0$, and $c = -9$, resulting in

$$x = \frac{-0 \pm \sqrt{0^2 - 4(4)(-9)}}{2(4)}$$

- Goal: to solve a quadratic by applying inverse operations.

Focusing on the three operations involved in the expression (squaring, multiplying by 4 and subtracting 9) a new structure emerges, one that can be emphasized by rewriting the expression as $4(x)^2 - 9$. With these operations identified as separate chunks, we can set this expression equal to zero and apply inverse operations in reverse order. This results in $4(x^2) = 9$, $x^2 = 9/4$, and finally

$$x = \pm\sqrt{\frac{9}{4}}$$

- Goal: to graph the quadratic.

We can see the symbolic representation $y = x^2$ as a base graph, with a vertical stretch of 4 and horizontal shift of 9 units down. Such transformations of the original $y = x^2$ might be symbolized as $y = 4(x^2) - 9$ and graphed as in Figure 2. With each of these four goals, the expression's structure was not an inherent feature of the symbolic notation. Instead, we were able to perceive the structure only when we recognized mathematical properties associated with the goals in the representation. It is through this connecting process that structure emerges—it exists for a student only when the student sees it. This stands in contrast to structure being an intrinsic attribute that exists in mathematical representations.

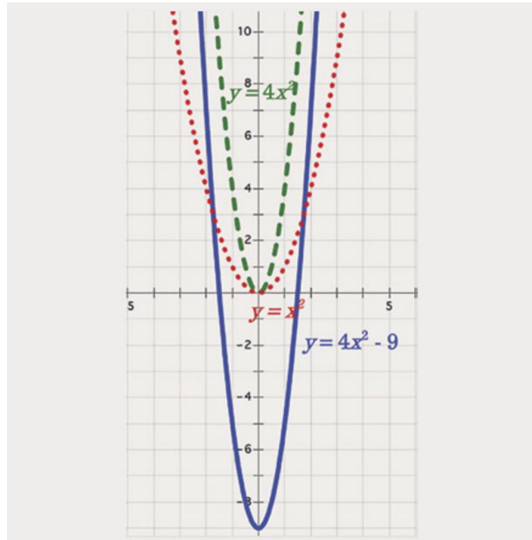


Figure 2: This graphical representation shows transformations associated with $y = x^2$.

To summarize, we might help students by thinking of structure as existing when we connect mathematical understanding to the mathematical representations, with a particular goal in mind. We now provide three ways that have helped our students look for and make use of structural reasoning.

Helping Students Develop a Structural Lens

Along with the role that representations and goals play as we look for and make use of structure, we find it useful to think of structure as a lens through which mathematics can be viewed. One must develop a structural lens just as one develops any productive habit and learn when and how to use it to be effective. This lens is similar to what has been referred to as *structural thinking* (Mason, Stephens, and Watson 2009), *structural sense* (Hoch 2003), or *structural reasoning* (Bishop et al. 2016). These terms suggest a disposition where one looks for, uses, and connects underlying mathematical properties in representations. In contrast to the way we learn a technique or a procedure (Mason et al.
2009), we must develop structural sense over time. This is an understanding that teachers must think about developing during the course of the entire year by repeatedly drawing attention to this practice. We build on the work of Hoch and Dreyfus (2005) by providing three components involved in structural reasoning (see Figure 3). We also provide examples to illustrate each of their meanings.

1. Recognizing equivalent or similar mathematical properties in different forms and representations

The first component of structural reasoning is the ability to *recognize equivalent or similar mathematical properties in different forms and across multiple representations*. Rather than rely on contextual characteristics, this skill involves connecting similar ideas that may be represented in multiple ways. For example, we teach the slope-intercept and point-slope as two distinctive forms for linear equations. Our teaching experiences suggest that students often do not see these forms as connected. By taking a structural lens, one can see both forms as instantiations of two properties that describe a line: a *fixed point* (e.g., an intercept in one form and a general point in the other) and a *direction* (e.g., the slope).

Seeing both the slope-intercept and point-slope forms as representations involving a fixed point and a direction can be highlighted through graphing. Although notationally $y = mx + b$ and $y = m(x - x_1) + y_1$ look very different, their equivalent structure becomes more apparent by connecting these forms to their graphical representation and asking the

1. Recognizing equivalent or similar mathematical properties in different forms and multiple representations.
2. (a) Seeing a mathematical expression (or parts of a mathematical expression) as an object as well as a process.
   (b) Decomposing (or chunking) algebraic expressions into a variety of sub-structures based on the context and goals at hand.
3. Making sense of appropriate manipulatives that productively uses the structure instead of automatically applying a set procedure.

Figure 3: Three components of structural reasoning involve recognizing, decomposing and making sense of properties, expressions and manipulations.

question, "For what value of x can we evaluate each expression so that a point on the line can easily be identified?" For a linear function in slope-intercept form $y = mx + b$, one answer is $x = 0$, producing $y = b$. This means that the line passes through the y-intercept (0, b). In the point-slope form $y = m(x - x_1) + y_1$, the choice of $x = x_1$ makes the first part of the expression equal to zero, leaving $y = y_1$. This means that the line passes through (x_1, y_1). By finding the x-value that readily yields a y-value, we can identify the coordinates of a point in each form. Consequently, students can understand a graphed line knowing a point and slope, whether that equation is written in slope-intercept form or point-slope form.

Additionally, the Common Core further emphasized the relationship between these two forms through the elevation of transformations. With this lens, the slope-intercept form $y = mx + b$ can be interpreted as a vertical shift of the line $y = mx$, and the point-slope equation $y = m(x - x_1) + y_1$ can be interpreted as a horizontal shift of x_1 and a vertical shift of y_1, meaning that $y = mx$ now passes through (x_1, y_1).

2. (a) Seeing expressions as objects as well as processes

The second two components of structural reasoning are interrelated, one being an understanding and the other an associated skill. The first is an understanding that enables students *to see a mathematical expression (or pieces of a mathematical expression) as a single object that can be operated on*. This interpretation contrasts seeing an expression as individual symbols combined by operations. Algebraic expressions can simultaneously represent the process of a computation and the object of that process (Sfard 1995). For example, the expression $x + 3$ can be interpreted as the *process* of adding three to an unknown quantity. It can also be interpreted as an object in and of itself, which is the result of three more than the quantity x. This difference can be emphasized contextually. For example, if the cost of a dinner is x dollars, and the tip is $3, from a process perspective, $x + 3$ is the process of adding three dollars to the cost of the dinner. From the object perspective, x + 3 would represent the total cost of dinner.

With numerical operations, the distinction between process and object is easier to see because we typically use a different symbol for the object that results from the process (i.e., the process 12 + 3 can be represented by the single object 15). With algebraic expressions, no alternative exists to highlight the resulting object, as the result of $x + 3$ is $x + 3$. It may be challenging for students to see expressions both as individual symbols combined by operations and as a single object (the result of these operations). Consequently, students often struggle with the property of closure, not seeing $x + 3$ as a viable answer (Tabach and

Friedlander 2008). Not accustomed to seeing arithmetic expressions as objects, students may feel compelled to simplify algebraic expressions incorrectly, adding unlike terms, such as writing $3x$ in place of $x + 3$.

(b) Chunking algebraic expressions into substructures

Once students are able to interpret an algebraic expression as an object, they have access to a different way of thinking. Students are able *to decompose algebraic expressions into a variety of substructures according to the context and goals at hand*. Take the equation $2(x - 4) = 10$. Students who see the left side of the equation as a set of operations are able to solve this equation by undoing the processes being applied to x in the reverse order (divide by 2 and add 4). In contrast, for those who are able to take a step back and see $x - 4$ as an object, the question then becomes what number, multiplied by 2, gives 10? This leads to an understanding that the object $x - 4$ must be equivalent to 5, and x is 9. Such an approach of viewing the expression as embedded chunks has been referred to as the "cover-up" method (Herscovics and Kieran 1980).

Cuoco, Goldenberg, and Mark (2010) refer to this component of structural reasoning as chunking. Chunking is often associated with factoring, but such an ability supports students in a wide variety of mathematical contexts. The understanding associated with chunking is critical when solving quadratics where factoring is necessary or with more challenging rational expressions (e.g., $11 - 50/(x - 2) = 6$). We can also use chunking when finding the domain and range of the function

$$f(x) = 8\sqrt{1 - \frac{25}{x^2}}.$$

We can interpret $f(x)$ as a series of function decompositions. This requires reflecting on the structure and identifying various symbolic pieces as separate, individual chunks. First, looking for the domain, imagine the radicand $g(x) = 1 - 25/x^2$ as a function. For the square root of $g(x)$ to be real, the output of $g(x)$ must be nonnegative. Likewise, by interpreting and analyzing $25/x^2$ as a new chunk or expression that is subtracted from 1, we can see this chunk must be less than or equal to 1 for the radicand to be nonnegative. Further decomposing $25/x^2$, we realize that the denominator x^2 must be larger than or equal to the numerator 25 for the fraction $25/x^2$ to be less than or equal to 1 and the output of $g(x)$ to be nonnegative. Therefore, x must be greater than or equal to 5 or less than or equal to –5. Similarly, with the range, we leverage the fact that x^2 must always be positive to reason that $25/x^2$ will also be positive. We can conclude that $1 - 25/x^2$ will be strictly less than 1. Therefore, the range will be less than $8\sqrt{1}$, which is equal to 8. The smallest value will be 0 because the radical must be nonnegative, which occurs when $25/x^2$ is equal to 1.

Although chunking may be a specific case or even the result of an object understanding of notation, we see these two components of structural reasoning as mutually supportive. As students develop the ability to take a step back and see algebraic expressions as objects, they are able to see decompositions of expressions as multiple pieces according to the goals and context. Likewise, encouraging students to identify different chunks within algebraic expressions leads to an understanding of algebraic expressions as an object. Although details of this relationship are beyond the scope of this article, we encourage others to explore what activities might support students to develop connections between chunking and process/object reasoning.

3. Using the lens of structure to make sense of appropriate manipulation
Finally, as students begin to possess the skills to decompose representations and chunk expressions into objects, they may engage in the last component of structural reasoning. This component requires students *to pause to examine the structure and decide whether one manipulation may simplify a problem more than another*. This contrasts with automatically applying a set procedure to solve a problem. Making sense of the next steps that take advantage of structure is difficult to develop, as demonstrated by Hoch and Dreyfus (2004). When college-bound juniors were asked to solve $1/4 - x/(x - 1) - x = 6 + 1/4 - x/(x - 1)$, close to 90 percent of them multiplied both sides of the equation by a common denominator to convert it into a linear equation, rather than observing that the expression $1/4 - x/(x - 1)$ occurs on both sides of the equation. Noting this similarity in structure can help students see the original equation as equivalent to $-x = 6$. Although all students in the study were exposed to the "substitution method" in solving quadratics (i.e., substituting u for $(x - 4)$ in the expression $2(x - 4)^2 - 5(x - 4) + 3$), very few applied this technique. This may be because students viewed such a method as just that, a specific technique, not an overall orientation that permeates their thinking.

In the Eye of the Beholder
Whether an elementary school student is asked to complete the equation 123 + 98 = 122 + ___; a middle school student, to solve for x in the inequality $|x - 3| > -4$; or a high school student, to use the quadratic formula to interpret the relationship of the roots to the vertex, structural reasoning is an important process and practice that shapes students' understanding of mathematics. With fear of stating the obvious, we note that we can only see what we look for. By taking a step back and looking for properties embedded in multiple mathematical representations, students can develop abilities to see expressions as both processes and objects, to chunk expressions into substructures, and to evaluate their next steps before automatically applying procedures. We find that thinking about structural reasoning as a lens for interpreting mathematics to be powerful for supporting students in learning mathematics.

The three qualities described here involved in looking for and making use of structural reasoning (see Figure 3) can be developed by teachers and used by students while problem solving. Although such an approach to mathematics is undoubtedly challenging to develop, focusing on structural thinking provides a powerful new way to reason mathematically. Because structure is in the eye of the beholder, teachers can play an important role in developing structural reasoning in students. As students practice looking for and using structural thinking across mathematical representations, they will begin to draw connections between previously compartmentalized topics. In this process of connecting mathematics, students will find greater enjoyment in developing their mathematical practices. We encourage others to further consider ways to develop structural reasoning in students.

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Student Efficacy Beliefs: What is the Impact of Group Work?

Beth Baldwin

Teaching secondary mathematics has its challenges, to be sure. One of the greatest challenges that I often face is student attitude towards mathematics. Throughout my years as a secondary mathematics teacher, I have become increasingly aware of students' attitudes towards mathematics, which seem to strongly impact their learning and work in my classroom. In particular, I have often observed that the more confident and growth-focused students are more engaged in their learning than students who have negative feelings towards their work. This, in turn, made me wonder: What could I do as a teacher to positively impact student attitude about their mathematical abilities?

I am certainly not the first to observe the way in which student beliefs, particularly negative ones, can impact learning. I have had many conversations with other educators over my decade of teaching that have had a similar tone: "Why does (student) seem so negative about their performance, despite having done so well?" This phenomenon has perplexed and frustrated teachers across other disciplines, but seems to be more concentrated in mathematics. Many students appear to enter the mathematics classroom already having formed strongly held beliefs about their abilities. This seems particularly true in secondary schools, as students have already had ample experience in what they have come to know as mathematics through their elementary and middle school years.

I have often observed that the more confident and growth-focused students are more engaged in their learning than students who have negative feelings towards their work.

This curiosity, and at times frustration, about students' beliefs led me to pursue a graduate degree in secondary mathematics education from Simon Fraser University. For my thesis work, I conducted research focusing on the relationship between self and group efficacy beliefs in secondary mathematics. Through this work, I have gained some insight into this relationship. This article aims to give an overview on efficacy beliefs, as well as to present my own findings and their possible classroom implications.

What is Efficacy?

All past experience with mathematics informs students' beliefs about their mathematical ability. This is more formally known as self-efficacy—a student's beliefs about his or her ability to perform a certain task. Self-efficacy was first introduced in psychology literature by Albert Bandura (1977) in a more general context, but has since been studied in connection with many areas. According to Bandura, there are four means of changing self-efficacy: personal mastery experience, vicarious experience, social feedback, and mood. Among these, personal experience (either positive or negative) is the most impactful (Bandura, 1977). This helps to explain why students in secondary mathematics have a more strongly held sense of belief than elementary students—they simply have more experience from which to draw conclusions. In short, Bandura's work shows that the more positive experiences a person has in a certain realm, the more likely they are to think positively about their abilities in this realm in the future.

The connection between self-efficacy and mathematics ability has been researched by many—so much so that a Mathematical Self-Efficacy Scale (MSES) was developed by Hackett and Betz (1989). This metric has individuals place themselves on a scale for different belief statements about their abilities in mathematics. The scale aims to measure an individual's efficacy beliefs, but does not indicate whether or not these beliefs accurately reflect the individual's ability. On the other hand, *calibration* is the notion of aligning self-efficacy beliefs with ability.

The calibration of self-efficacy has been studied most effectively when content is specified—for instance, particular mathematics topics, or even specific questions. If you ask a student if they are 'good at mathematics,' that is different than asking a student if they are 'good at algebra.' Even more specifically, we can ask a student if they are good at 'solving the equation $x + 5 = 2$'. As the task being assigned to students gets more and more specific, students can narrow in on particular past experiences to inform their self-efficacy beliefs. Pajares and Miller (1997) showed just this—that specific content showed more calibrated self-efficacy beliefs. Additionally, context plays a significant role in efficacy beliefs. Whether a student is asked how they will do in a group setting, on homework, or on a test has also been shown to impact efficacy beliefs (Pajares & Miller, 1997).

As more and more mathematics educators shift towards group activities in the classroom, I became interested in students' self-efficacy beliefs while engaging in group work. How do students feel when working in groups, as compared to working alone? Do their efficacy beliefs change? However, before I discuss my research in this area and the results, I would like to provide a brief overview of the group work model at the center of my work.

The Thinking Classroom Model

You may already be familiar with Peter Liljedahl's *thinking classroom* (2016) model, as it has been growing in popularity among mathematics educators over the past several years. I was introduced to this approach by way of my graduate studies program. In January of 2016 I took a course with Dr. Liljedahl at Simon Fraser University and was immediately captivated by his approach to the teaching and learning of mathematics. There are two strategies that Liljedahl argues should be implemented first, as they have been proven to be among the most effective at engaging learners:

1) Have students complete (some of) their daily work on a vertical non-permanent surface (VNPS).

2) Have students work in visibly random groups (VRG).

In my own classroom, these strategies were implemented as follows. Every class began with me shuffling a deck of cards and students selecting a card from the deck. It was important that students were able to see me shuffle the deck so that they were aware of the randomization. By visibly randomizing groups, Liljedahl has shown that students know there is no hidden agenda to their grouping and over time, students become more agreeable to work in any group. Their selected card would be their group assignment for that class (all "1"s formed a group, all "2"s formed a group, etc.). After card selection, I would provide an oral description of a problem or task for students to work on with their group. Rarely were there visuals or written descriptions of any kind, but if these were necessary, I would make a photocopy for each group to have as they worked. Liljedahl argues that oral instructions encourage prompt group discussion and collaboration. Once the problem or task was assigned, students would go to their stations, which were marked by numbers around the classroom corresponding to the cards they drew. They would then begin working on the task or problem on whiteboards situated around the perimeter of the classroom, with only one marker provided per group to promote discussion. This work would continue for anywhere from ten minutes to over an hour, depending on the students' age, engagement, and the depth of the task.

Group problem-solving tasks have been shown to increase student engagement and collaboration (Liljedahl, 2016). In particular, Liljedahl argues that VRGs and VNPS are among the most effective ways to engage learners in mathematics classrooms. It has been interesting to observe attitude shifts in my own students when they are working together—students seem more positive in their approach to mathematics. My observations have led me to believe that, in general, students will feel more confident in their mathematical abilities in groups than individually.

Study

My study followed 104 students in grades 11 and 12 for three months during the spring of 2017. All of the students were taking part in a thinking classroom. The students were initially given a questionnaire that consisted of four parts: Parts 1 and 2 looked at general mathematics efficacy beliefs in the context of individual and group work, respectively. This part asked students to agree or disagree on a five-point scale on statements such as "I am good at mathematics" and "I believe I can understand the content in a mathematics course." Parts 3 and 4 looked at more specific mathematics content beliefs in the context of individual and group work, respectively. These parts asked students to state how well they thought they would perform on a five point scale, from very poorly to very well, on topics such as "solving an equation algebraically," "working without a calculator," and "word problems". This questionnaire was administered twice over the course of six weeks. For those interested, the questionnaire is available in full at the Simon Fraser University library in the appendix of my thesis (Baldwin, 2018).

After analyzing the questionnaire, I decided to interview some of the students who showed higher variation in their self and group efficacy scores. The purpose of the interviews was to gain further insight as to the rationales behind the students' efficacy beliefs. After interviewing the students, not only was I able to discern why they believed what they did, I also noticed that there were three distinct groups within those interviewed.

Findings

As I had expected, the questionnaire showed that the majority of students (approximately 70%) reported more positive group-efficacy than self-efficacy. This makes sense: the notion that "more brains are better than one" is an old adage for a reason. The interviewees in this first group echoed this notion and stated that they felt they would benefit from others' ideas. These students reported that they would turn to group members in situations where they felt stuck or uncertain about a problem or task. Although I did not interview all students surveyed, it is my hunch that this is what the majority of students, or more generally people, would say. These students gave particular examples of times when they were stuck and another student had offered a suggestion that moved the group forward. It was clear that they felt they benefitted from others' ideas and would be less successful without them.

The second group was comprised of students that indicated more positive self-efficacy beliefs than group-efficacy beliefs. These students preferred, and thought they would perform better, working alone. The students provided several reasons, but most notably they all thought that a group would make them less efficient, hindering their progress. Group work is sometimes time-consuming and these students felt it could be inefficient. These students were those with strong mathematical abilities and tended to perform well on assessments. Additionally, they frequently had keen mathematical insights and may not have benefitted from a group as much as other students. One of these students also commented on being confused working in groups as she sometimes had to explain her thinking and was not always able to clearly do so.

There was a third group of students that I did not anticipate. This group also had more positive group efficacy beliefs, but the reasoning they provided differed slightly from the first group. This subgroup felt that they would feel more confident in a group because they were generally not confident in their own ideas. That is, these students may have been perfectly capable of completing a task individually, but they had significant doubt in their work. Their reasoning was that the addition of a group could serve as a means of checking their thinking and reaffirm any ideas they may have. These students all reported feeling anxious about mathematics. Interestingly, they were all female.

It should be stated that this research was conducted in a relatively affluent area in West Vancouver and all students surveyed were enrolled in more academic courses: Pre-Calculus and International Baccalaureate (IB). Students enrolled in a less academic or abstract mathematics course—for example, Foundations of Mathematics or Apprenticeship & Workplace – would likely have different feelings towards their efficacy. It would be interesting to extend this research to a larger variety of students.

Practical Takeaways

After observing some of the short-term impacts of group work on mathematics efficacy beliefs, I wonder what the long-term impact might be. I have a hunch that more experience with group work (from none) would have positive impacts on both group- and self-efficacy beliefs. This is because the more positive group efficacy beliefs would be a means of changing self-efficacy beliefs, as outlined by Bandura (1977). In particular, students would be privy to three of four means of changing self-efficacy: mastery experience (in the group), vicarious experience of their classmates, and positive feedback from their group. Even if it was another student who had specific insights or 'success' on a problem, this vicarious experience may lead students to feel as though they too were a part of this success, regardless of how much they were directly involved. Having experienced success in mathematics in the context of a group, I have a hunch that these experiences could positively impact self-efficacy beliefs, and possibly attitudes towards mathematics.

Even if it was another student who had specific insights or 'success' on a problem, this vicarious experience may lead students to feel as though they too were a part of this success.

I have anecdotal experience that this is the case. Just this past fall, I recall working with several students in my Math 9 classroom who I had observed to have low mathematical self-efficacy. These students had not been involved in my formal research, but they often remarked that they were "bad at math" and that they thought they would not do well in our course. After working in a thinking classroom model for only a few short months, I noticed fewer of these statements and that these same students seemed more positive towards their work. Of course, I cannot make a claim that this is directly because of their experiences with group work; however, it is definitely a possibility, as hypothesized above. Research that follows efficacy beliefs from the start of the school year in a thinking classroom could look for changes in self-efficacy beliefs.

Nevertheless, the impact of the thinking classroom model on efficacy beliefs cannot be ignored. Student confidence, which is a more generalized and context-less form of self-efficacy did seem to improve. There is a shift towards a warmer and more positive classroom environment during group work in which collaboration and communication are paramount—in my own classroom, this came by way of the thinking classroom model. This positivity could only serve to improve efficacy beliefs. My hope is that improved group-efficacy beliefs would give students more resilience when working individually and lead to more positive self-efficacy beliefs.

Here in British Columbia, there has recently been a shift towards more emphasis on group work in the mathematics classroom, which naturally takes away from individual work time. We have a new mathematics curriculum from Kindergarten through Grade 12 that outlines more focus on skills as opposed to content standards. All secondary mathematics courses now have a set of 'curricular competencies' that outline a more general set of underlying skills that students should be developing throughout the course. Among these competencies are two that can be facilitated particularly effectively in the context of group work: 'communicating and representing' and 'connecting and reflecting' (more details available at https://curriculum.gov.bc.ca/curriculum/mathematics). The documents also mention the importance of classroom discourse, as well as the ability to make connections between mathematical concepts. If it is generally true that students report more positive group than self-efficacy beliefs, then this emphasis on group work may be a more positive way of learning and experiencing mathematics for the majority of students within the framework of the curriculum.

> After initially having positive experiences in a group, students can gradually move towards more individual work, should we so choose.

Despite the benefits of group work, we cannot underestimate the value of individual work and practice time. I feel that after initially having positive experiences by means of group activities and problem solving, students can gradually move towards more individual work, should we so choose. This would theoretically give most students the opportunity to experience positive efficacy and would likely lead to positive associations with mathematics in general. After positive experiences with group work, my hope is that this positivity would impact their self-efficacy beliefs for the better as well.

However, if we are seeking to improve student self-efficacy beliefs and attitude in mathematics, we also need to have students realize themselves that their beliefs may be impacting their learning. Simply by discussing what efficacy beliefs are and having students become more aware of their self-talk, we as teachers can help students move forward with new, positive experiences. Carol Dweck (2006) is well known for her research on the growth mindset—the belief that we are able to learn and grow, and if we are currently unable of accomplishing a certain task, it does not mean that we will never be able to. By educating ourselves and our students about what efficacy is, and providing opportunities to develop more positive beliefs around them, we are helping to create positive learning environments that foster empowered students.

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Beth Baldwin has been teaching secondary mathematics in British Columbia since 2008. She recently completed her Master's thesis from Simon Fraser University, where she examined the relationship between self and group efficacy in secondary mathematics. Beth is experimenting with the thinking classroom model and is particularly interested in the psychological and emotional impacts on students. Beth is currently learning the new joys and challenges of parenthood as she is on maternity leave.

Encouraging Mathematical Habits of Mind: Puzzles and Games for the Classroom

Streamers & Kakurasu Puzzles

Susan Milner

Mathematical logic games are a marvellous way of developing students' reasoning abilities. I visit hundreds of classrooms a year, sharing games with students of all ages, and am always struck by how much fun students have and how surprised their teachers are by the level of thinking the students demonstrate.

I have discussed a number of these games, including some for primary students, in previous editions of *The Variable.*[1] Here are two more that have proven to be very popular in intermediate through secondary classrooms.

Streamers

Recommended for grades 6-12, but has also been very successful in grade 4/5 classes where students already have experience with math/logic games. Manipulatives are required.

I introduce the game by creating a grid on the board using magnetic pieces (see Figure 1). Then, I ask the class:

"What do you notice about this solved puzzle?"

Cover up the answers below until you've had a chance to think!

Students come up with a variety of observations, including the following:

- There are four shapes.
- Each shape comes in four colours.
- Each colour appears once in every row and every column.
- Each shape can be connected to other pieces of the same shape by a one-step horizontal, vertical, or diagonal line.

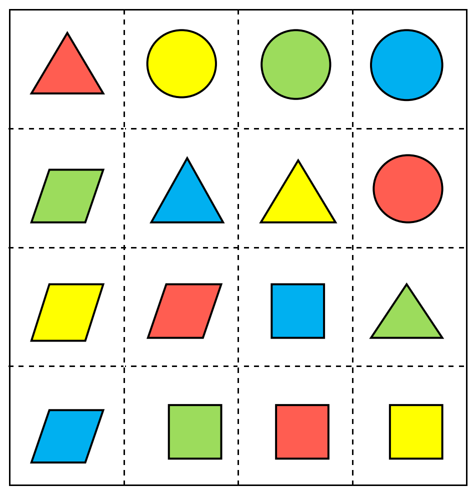


Figure 1: A solved Streamers puzzle

We then solve the next puzzle together. I ask students to make only one suggestion, not a chain of them, so that everyone can follow each step in the logic. The single most important thing is to look for something we know *must* be true. No guessing, as that usually results in a mess. Knowing the difference between what must be the case and what might be only possible is, in fact, central to solving all logic puzzles – and very important in solving any type of mathematical problem.

[1] See issue *1*(3) of *The Variable* for an introduction to Domino puzzles and Rectangles, *1*(5) for Hidato and Latin Squares, *1*(8) for SET®, and *2*(2) for Difference Triangles.

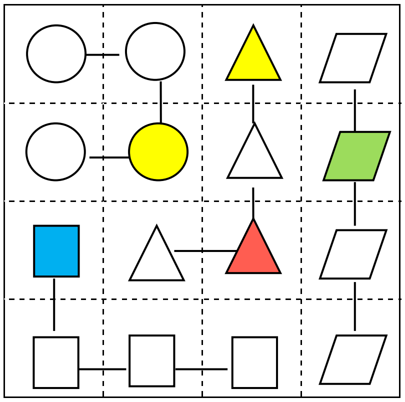
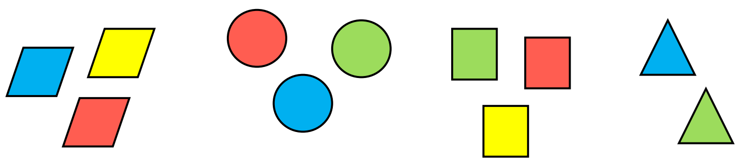


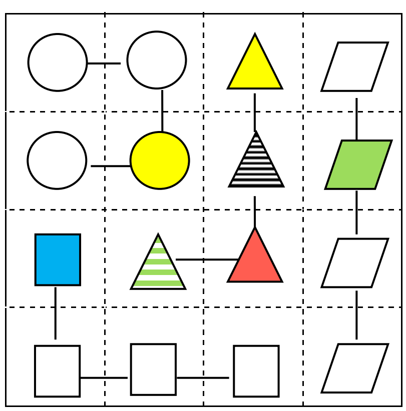
Figure 2: An easy Streamers puzzle

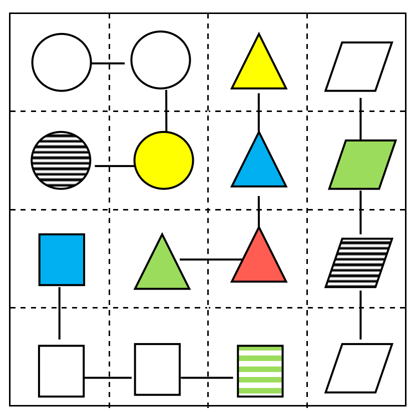
Can you find a place where you know *for sure* a particular piece must go?

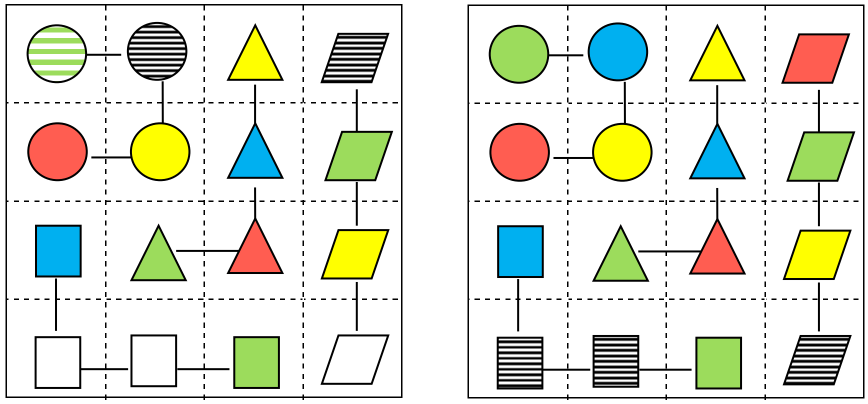
These are the pieces that you have left to play with:



As in all of these kinds of math/logic puzzles, there may be several good places to start, or several good subsequent moves at any stage. In the interests of saving space, I will present a logical sequence of moves, two or three pieces at a time. Each new piece is shown as striped. The coloured striped piece goes first. Then, think about what colour the black striped piece(s) should be. Readers will probably derive more enjoyment from covering up each step until they have made their own decision, rather than skimming over my solution.



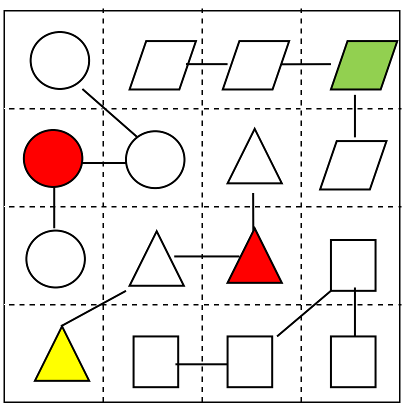


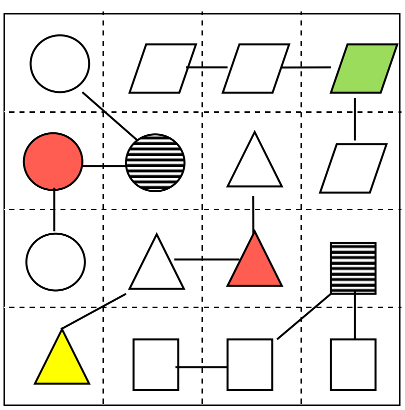


Figures 3-6: A solution sequence for the Streamers puzzle in Figure 2

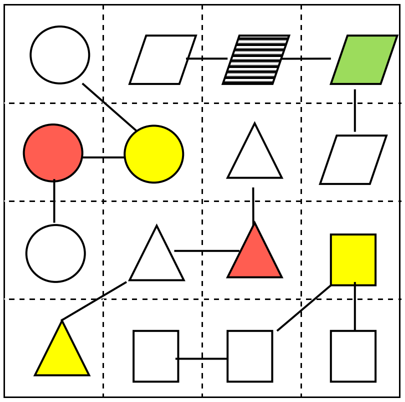
Once we have solved a puzzle as a class, every student gets a bag of coloured pieces and the first printed puzzle. Students move on to the next puzzle once an adult has checked their solution to the current one. The puzzles gradually get more challenging. As students get comfortable with the rules, they need less oversight.

A slightly more challenging Streamers puzzle appears below, in order to demonstrate more complex reasoning.





The shaded figures are the only places in the circle streamer and the squares streamer where it is possible to place a yellow piece.



There is only one place for the last yellow piece to go. From this point, the puzzle is easy to solve!

If you are interested in introducing your class to Streamers, you will find printable puzzles and a template for the pieces at http://susansmathgamesca.ipage.com/streamers/. I made my pieces out of craft foam, but card stock would also work.

Streamers is a modified version of Strimko, a numbers-only puzzle created by the Grabarchuk family in 2008. I have found that puzzles incorporating shapes and colours are more easily accessible for many people, so I created Streamers in the hope that some teachers and students might be drawn to Strimko after trying Streamers. Books and apps featuring Strimko puzzles in different sizes can be easily sourced on-line.

Kakurasu
Recommended for grades 7-12, but has also been successful in grade 4-6 classes where students already have experience with math/logic games. This is a paper and pencil puzzle that does not require manipulatives.

It's a bit tricky at first, but is well worth the effort!

What do you notice about this solved puzzle (Figure 7)? (A row and a column have been shaded in order to give you a hint.)

	1	2	3	4	
1	✗	✓	✓	✗	**5**
2	✓	✓	✓	✗	**6**
3	✗	✗	✓	✓	**7**
4	✓	✓	✗	✗	**3**
	6	**7**	**6**	**3**	

Figure 7: A solved Kakurasu puzzle

Here is another solved puzzle (Figure 8), so that you can look for commonalities.

	1	2	3	4	
1	✓	✓	✓	✓	**10**
2	✖	✓	✖	✖	**2**
3	✖	✖	✓	✓	**7**
4	✓	✖	✓	✗	**4**
	5	**3**	**8**	**4**	

Figure 8: Another solved Kakurasu puzzle

If you think you know what's going on, fill in the blanks on the right and bottom of the following puzzle (Figure 9).

	1	2	3	4	
1	✖	✖	✖	✓	___
2	✓	✓	✖	✖	___
3	✖	✓	✓	✖	___
4	✖	✓	✓	✓	___
	___	___	___	___	

Figure 9

A row and a column have again been shaded, in order to give you something to focus on. The answer is below (Figure 10).

	1	2	3	4	
1	✖	✖	✖	✓	**4**
2	✓	✓	✖	✖	**3**
3	✖	✓	✓	✗	**5**
4	✖	✓	✓	✓	**9**
	2	**9**	**7**	**5**	

Figure 10

By the time we've discussed at least one solved puzzle and filled in the right side and bottom of another, many students are able to deduce the rules.

The rules of Kakurasu:

- Our goal is to place the ✓ marks where they will produce the given totals on the right and the bottom of the grid.
- The targets on the right give the totals for the rows.
- The targets across the bottom give the totals for the columns.
- The numbers across the top and on the left give the values that contribute to the totals.
- Marking a square with a ✓ means that square's value gets added to both the row's total and the column's total.
- ✗ represents a box that cannot be used.

Guessing or relying on "intuition" might work on smaller puzzles, but it doesn't work for bigger ones. It is a good idea to deliberately practice your logic on the easier puzzles before moving on to harder ones.

In the classroom, we would solve the easy 4x4 puzzle in Figure 11 as a group, with students taking turns to make a single suggestion. It can take a bit of time for everyone to feel comfortable with the direction in which to read the numbers for the given targets.

You might like to try it yourself before reading the discussion below!

While there is only one correct solution to a puzzle, there are usually several ways to get to that solution. Here is one sequence of reasoning:

	1	2	3	4	
1					**7**
2					**5**
3					**7**
4					**4**
	4	**2**	**10**	**4**	

Figure 11: An easy Kakurasu puzzle

Consider the targets of 10 and 2. There is only one way to get each of these (see Figures 12 and 13).

	1	2	3	4	
1			✓		**7**
2			✓		**5**
3			✓		**7**
4			✓		**4**
	4	**2**	**10**	**4**	

Figures 12

	1	2	3	4	
1		✗	✓		**7**
2		✓	✓		**5**
3		✗	✓		**7**
4		✗	✓		**4**
	4	**2**	**10**	**4**	

Figure 13

Now we can consider the rows.

We have reached the target of 5, so we finish the second row by placing the Xs (Figure 14).

	1	2	3	4	
1		✗	✓		**7**
2	✗	✓	✓	✗	**5**
3		✗	✓		**7**
4	✓	✗	✓	✗	**4**
	4	**2**	**10**	**4**	

Figure 14

In the bottom row we already have 3, so we just need 1 to get 1 + 3 = 4.

Now the puzzle can be completed using either the remaining rows or columns (Figure 15). Here, I have used the columns, so we know that we have solved the puzzle because the final row targets are also met.

	1	2	3	4	
1	✗	✗	✓	✓	**7**
2	✗	✓	✓	✗	**5**
3	✗	✗	✓	✓	**7**
4	✓	✗	✓	✗	**4**
	4	**2**	**10**	**4**	

Here is a slightly harder 4 x 4 puzzle, with some suggestions (Figure 16):

	1	2	3	4	
1					**7**
2					**5**
3					**4**
4					**8**
	5	**2**	**7**	**8**	

The only way to reach a target of 2 is to use 2 itself.

The only way to reach 8 is to subtract 2 from 10.

	1	2	3	4	
1		✗		✓	**7**
2		✓		✗	**5**
3		✗		✓	**4**
4	✓	✗	✓	✓	**8**
	5	**2**	**7**	**8**	

The rest of the puzzle follows quite easily.

Some general strategies for solving Kakurasu puzzles:

Small-ish numbers

- Anywhere that there is a 1 or a 2 is a good place to start, as there is only one way to make either of those work. This is true no matter how large the puzzle.

- While there are two ways to reach 3, we know for sure that no number larger than 3 can be used. This is true no matter how large the puzzle.

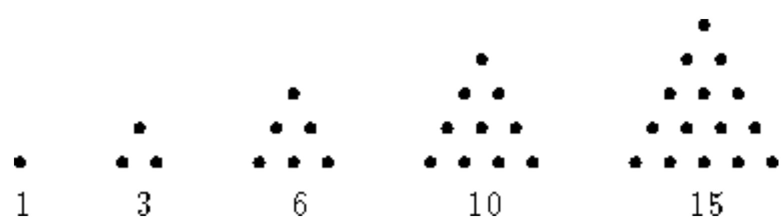
1	2	3	4	
			✗	**3**

- What can we conclude about 4? There are only two ways to reach 4, either $4 = 4$ or $4 = 1 + 3$. There is no way to use 2, so we can get rid of that possibility.

1	2	3	4	
	✗			**4**

Large-ish numbers

- Other good places to start a puzzle involve the appropriate *triangular number*. The n^{th} triangular number is the sum of the first n counting numbers. For example, 10 is the 4^{th} triangular number because we can form a triangle from 10 dots, with one more dot in each row than in the previous one.



The first five triangular numbers

Therefore, in a 4x4 puzzle, 10 is a good place to start. So are 9 and 8, as there is only one way to get each of them, 9=10-1 and 8=10-2. On the other hand, 7 does not have a unique decomposition, because 10-3 can be found by deleting the 3 or by deleting 1 and 2.

- We can, however, conclude something about 7 in a 4x4 puzzle:

1	2	3	4	
			✓	**7**

Larger puzzles give us more to think about.

- In a 5x5 puzzle, 10 is no longer a good place to start. The most useful large numbers here are 15, 14, and 13.
- In a 6x6 puzzle, we need the 6^{th} triangular number, which is 21.
- Subtraction is often more useful (and quicker) than addition.
- You might find it useful to note down how many you have left to reach in a column or row. For example, suppose that in a 6x6 puzzle you have reached this point:

1	2	3	4	5	6	
		✓	✓			**13**

In order to remind yourself of the new target, you might want to write:

1	2	3	4	5	6	
		✓	✓			**13** (6)

Here are a few slightly harder puzzles for you to try. Some are missing a few target numbers, but solutions are still unique.

	1	2	3	4	5	
1						**7**
2						**9**
3						**9**
4						**7**
5						**9**
	10	**4**	**11**	**1**	**14**	

	1	2	3	4	
1					**?**
2					**5**
3					**6**
4					**?**
	4	**6**	**5**	**4**	

	1	2	3	4	5	
1						**?**
2						**3**
3						**11**
4						**?**
5						**8**
	14	**5**	**13**	**9**	**3**	

	1	2	3	4	5	6	
1							**10**
2							**7**
3							**5**
4							**8**
5							**17**
6							**9**
	16	**12**	**12**	**5**	**12**	**9**	

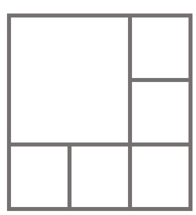
Two good sources for online Kakurasu puzzles are Brainbashers and Otto Janko's website.

Susan Milner taught post-secondary mathematics in British Columbia for 29 years. For eleven of those years she organised the University of the Fraser Valley's high school math contest – her favourite part was coming up with post-contest activities for the participants. In 2009 she started Math Mania evenings for local youngsters, parents, and teachers. Now retired and living in Nelson, BC, she shares her math/logic games with students all over BC, as a volunteer for Science World. She also gives professional development workshops. Lately she has been giving Brain Games classes for the Nelson Learning in Retirement program, which is a whole lot of fun. In 2014 she was awarded the Pacific Institute for the Mathematical Sciences (PIMS) Education Prize.

Problems to Ponder

Square Dissection

A number *N* is called 'nice' if a square can be dissected into *N* nonoverlapping squares. For example, as the following figure shows, 6 is a 'nice' number:



Which numbers are 'nice'?

Source: Mason, J., Burton, L, & Stacey, K. (1985). *Thinking mathematically*. Essex, England: Prentice Hall.

In this column, you'll find information about upcoming math education-related workshops, conferences, and other events. Some events fill up fast, so don't delay signing up! For more information about a particular event or to register, follow the link provided below the description. If you know about an upcoming event that should be on our list, please contact us at thevariable@smts.ca.

Within Saskatchewan

Technology in Mathematics Foundations and Pre-Calculus
March 6, 2020
Weyburn, SK

Technology is a tool that allows students to understand senior mathematics in a deeper way. This workshop is designed to have math foundations and pre-calculus teachers experience a variety of technology tools that allow students to represent and visualize mathematics concepts. Tools highlighted are useful for students to explore, learn, communicate, collaborate and practice, in order to enhance their understanding of mathematics in secondary mathematics.

More information at https://www.stf.sk.ca/professional-resources/events-calendar/technology-mathematics-foundations-and-pre-calculus

Saskatchewan IT Summit
May 4-5, 2020
Saskatoon, SK

The summit will create opportunities to:

- Explore exemplary practices for teaching and learning with technology to support the actualization of Saskatchewan curricula.
- Share best practices in network infrastructures and centralized technologies that support student learning through technology use in schools and school divisions.
- Promote professional dialogue that fosters effective teaching and learning with technology.
- Celebrate, support and encourage partnerships and networks of support.

More information at https://www.stf.sk.ca/professional-resources/events-calendar/saskatchewan-it-summit

Accreditation Initial Seminar
March 5-6 & 26-27, 2020
Saskatoon, SK
Presented by the Saskatchewan Professional Development Unit

Accreditation seminars are offered to enable qualified teachers to become accredited. Accreditation is the process by which qualified teachers are granted the responsibility of determining the final mark or standing of the students in a specified Grade 12 (level 30) subject or subjects. The Accreditation seminar provides an opportunity for teachers to challenge, extend, enhance and renew their professional experience with an emphasis on assessment and evaluation. Participation in this seminar results in partial fulfilment of the requirements for accreditation in accordance with the Ministry of Education's publication *Accreditation (Initial and Renewal): Policies and Procedures* (2017).

More information at www.stf.sk.ca/professional-resources/events-calendar/accreditation-seminar-initial

Accreditation Renewal/Second Seminar
March 5-6, 2020
Saskatoon, SK
Presented by the Saskatchewan Professional Development Unit

More information at https://www.stf.sk.ca/professional-resources/events-calendar/accreditation-renewalsecond-seminar

Beyond Saskatchewan

NCTM Centennial Meeting and Exposition
April 1-4, 2020
Chicago, IL
Presented by the National Council for Mathematics Teachers

NCTM turns 100 in 2020. Join thousands of math education professionals in Chicago as we celebrate at the Centennial Annual Meeting & Exposition. In addition to compelling sessions, networking opportunities, and valuable content, there will be special events and surprises to mark the occasion. Whether you're a PK to Grade 12 classroom teacher, math coach, administrator, math teacher educator, preservice teacher, or math specialist, you will want to join us in Chicago as NCTM starts its second century. Something like this only happens every 100 years!

More information at https://www.nctm.org/100.aspx

OAME 2020 Annual Conference
May 7-8, 2020
Oshawa, ON
Presented by the Ontario Association for Mathematics Education

The Ontario Association for Mathematics Education (OAME) is an organization for professionals interested in mathematics education. Our mission is to promote excellence in mathematics education throughout the province of Ontario.

OAME hosts an annual conference where educators have the opportunity to hear from keynote and featured speakers, attend workshops and networking events, and explore the latest resources available from exhibitors. Over 1200 educators attended the 2019 conference in Ottawa.

The theme of OAME 2020 is "In Focus", centering around balance, with an emphasis on:

- Well-Being (math anxiety of teachers and/or students, mental health, community engagement, mindfulness)
- Equity (Culturally Relevant Pedagogy, social justice, student voice, inclusion)
- Balanced Mathematics (Teaching & Learning: implementation, communication, technology, differentiation)
- Balanced Assessment (differentiation, diagnostic, summative, formative)
- Leadership (mentoring, coaching, collaborating, administration)

More information at https://sites.google.com/oame.on.ca/oame2020/home

Online Workshops

Education Week Math Webinars
Once a month, Education Weekly has a webinar focusing on math. They also host their previous webinars on this site. Previous webinars include Formative Assessment, Dynamic vs. Static Assessment, Productive Struggling, and Differentiation.

More information at www.edweek.org/ew/marketplace/webinars/webinars.html

Global Math Department Webinar Conferences
The Global Math Department is a group of math teachers that organizes weekly webinars and a weekly newsletter to let people know about the great stuff happening in the math-Twitter-blogosphere and in other places. Webinar Conferences are presented every Tuesday evening at 9 pm Eastern. In addition to watching the weekly live stream, you can check the topic of next week's conference and watch any recording from the archive.

More information at www.bigmarker.com/communities/GlobalMathDept/conferences

Problems to Ponder

Court Intrigue
A stranger asks you to shuffle an ordinary deck of cards and then cut it into three heaps. He'll bet you $20 that at least one of the topmost cards is a king, queen, or jack. Should you take the bet?

Source: O'Shea, O. (2016). *The call of the primes: Surprising patterns, peculiar puzzles, and other marvels of mathematics.* Amherst, NY: Prometheus.

Tangents

Extracurricular Opportunities for Students

This column highlights local and national extracurricular opportunities for K-12 students interested in mathematics, including collaborative, individual, online, and in-person challenges, contests, and camps. For dates, registration procedures and applications, and other information about the contests listed, please head to the contest websites, included below the descriptions. If we have missed an event that should be on our list, please contact us at thevariable@smts.ca.

If you are looking for contests available in other provinces, head to the Canadian Mathematical Society website (cms.math.ca/Competitions/othercanadian). The CMS also maintains a list of resources for students who are looking to build their problem-solving skills and succeed in competitions: see cms.math.ca/Competitions/problemsolving.

Canadian Math Kangaroo Contest
Written in March

The purpose of this competition is to introduce youngsters from Grade 1 to Grade 12 to math challenges in a fun and enjoyable way, thus inspiring their further interest and advancement in mathematics. The competition is held yearly in more than 50 Canadian cities. Students may choose to participate in English or in French.

More information at https://mathkangaroo.ca

Canadian Team Mathematics Contest
Written in April
Presented by the Centre for Education in Mathematics and Computing (University of Waterloo)

The Canadian Team Mathematics Contest (CTMC) is a fun and challenging competition for teams of 6 secondary school students in any combination of grades. One teacher and groups of six students participate at their own school over 3 consecutive hours. The curriculum and level of difficulty of the questions will vary. Junior students will be able to make significant contributions but teams without any senior students may have difficulty completing all the problems.

More information at www.cemc.uwaterloo.ca/contests/ctmc.html

Caribou Mathematics Competition
Held six times throughout the school year

The Caribou Mathematics Competition is a worldwide online contest that is held six times throughout the school year. Each of these days, five contests are offered, one for each of the grade levels 3/4, 5/6, 7/8, 9/10 and 11/12 and each one in English, French and Persian. The Caribou Cup is the series of all Caribou Contests in one school year. Each student's ranking in the Caribou Cup is determined by their performance in their best 5 of 6 contests through the school year.

More information at cariboutests.com

Euclid Mathematics Contest
Written in April
Presented by the Centre for Education in Mathematics and Computing (University of Waterloo)

The Euclid Mathematics Contest is an opportunity for students in their final year of secondary school and motivated students in lower grades to have fun and to develop their mathematical problem solving ability. Most of the problems are based on the mathematical curriculum up to and including the final year of secondary school. Most of the problems are based on curricula up to and including the final year of secondary school. Some content might require students to extend their knowledge and the best way to familiarize oneself with commonly appearing topics is to practice using past contests.

More information at www.cemc.uwaterloo.ca/contests/euclid.html

Fryer, Galois, and Hypatia Mathematics Contests
Written in April
Presented by the Centre for Education in Mathematics and Computing (University of Waterloo)

The Fryer, Galois and Hypatia Math Contests are an opportunity for students to write a full-solution contest. They are fun way to develop mathematical problem solving skills through a written mathematical activity. For students in Grades 9 (Fryer), 10 (Galois) and 11 (Hypatia). Questions are based on curriculum common to all Canadian provinces. Rather than testing content, most of the contest problems test logical thinking and mathematical problem solving.

More information at www.cemc.uwaterloo.ca/contests/fgh.html

Gauss Mathematics Contests
Written in May
Presented by the Centre for Education in Mathematics and Computing (University of Waterloo)

The Gauss Contests are an opportunity for students in Grades 7 and 8, and interested students from lower grades, to have fun and to develop their mathematical problem solving ability. Questions are based on curriculum common to all Canadian provinces. The Grade 7 contest and Grade 8 contest is written by individuals and may be organized and run by an individual school, by a secondary school for feeder schools, or on a board-wide basis.

More information at www.cemc.uwaterloo.ca/contests/gauss.html

Opti-Math
Written in March
Presented by the Groupe des responsables en mathématique au secondaire

A French-language mathematics challenge for secondary students who would like to exercise and develop their problem-solving skills.

Les Concours Opti-Math et Opti-Math + sont des Concours nationaux de mathématique qui s'adressent à tous les élèves du niveau secondaire (12 à 18 ans) provenant des écoles du Québec et du Canada francophone. Ils visent à encourager la pratique de la résolution de problèmes dans un esprit ludique et à démystifier, auprès des jeunes, les modes de pensée qui caractérisent la mathématique. Le principal objectif des Concours est de favoriser la participation bien avant la performance. La devise n'est pas : « que le meilleur gagne » mais bien « que le plus grand nombre participe et s'améliore en résolution de problèmes ».

More information at www.optimath.ca/index.html

Pascal, Cayley, and Fermat Contests
Written in February
Presented by the Centre for Education in Mathematics and Computing (University of Waterloo)

The Pascal, Cayley and Fermat Contests are an opportunity for students in Grades 9 (Fryer), 10 (Galois)m and 11 (Hypatia) to have fun and to develop their mathematical problem solving ability. Early questions require only concepts found in the curriculum common to all provinces. The last few questions are designed to test ingenuity and insight. Rather than testing content, most of the contest problems test logical thinking and mathematical problem solving.

More information at www.cemc.uwaterloo.ca/contests/pcf.html

The Virtual Mathematical Marathon
Supported by the Canadian National Science and Engineering Research Council

The virtual Mathematical Marathon has been developed by an international team of mathematicians, mathematics educators, and computer science specialists with the help of the Canadian National Science and Engineering Research Council.

The main activity is a competition allowing students to enjoy solving challenging mathematical problems all year around. Students can join the game at any time and at no cost, simply creating an individual profile with an individual username and a password. Available in French.

More information at www8.umoncton.ca/umcm-mmv/index.php

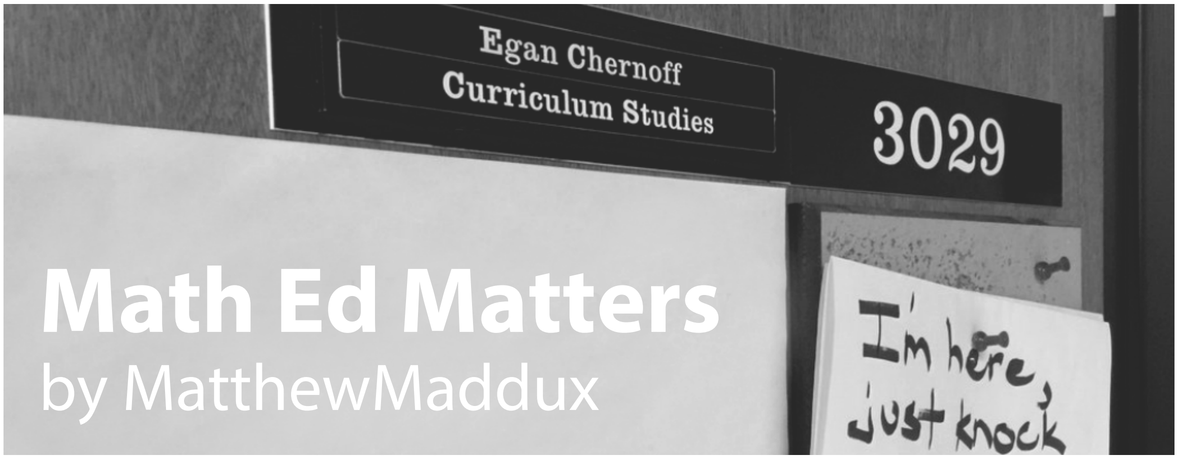
Problems to Ponder

Sums of Three Cubes
Both 11 and 12 can be written as the sum of the cubes of three integers:

$$11 = 3^3 + (-2)^3 + (-2)^3 \qquad 12 = 7^3 + 10^3 + (-11)^3$$

Which of the numbers from 1-100 can be written as the sum of the cubes of three integers?

Source: https://mei.org.uk/?section=resources&page=month_item



Math Ed Matters by MatthewMaddux *is a column telling slightly bent, untold, true stories of mathematics teaching and learning.*

Semicircular Reasoning in the Math Class: It's Not a Teaching Strategy Because It's Not

Egan J Chernoff
egan.chernoff@usask.ca

In a previous article[1], I defined the notion of a *mathematical abhorithm* as an abhorrent mathematical algorithm. In that article, I deemed an algorithm "abhorrent" if the mathematical algorithm (i.e., the set of rules used to correctly solve a mathematics problem) has no mathematical basis, ignores any underlying mathematical basis, or if the link between the abhorithm and any mathematical basis is not adequately taught. I also noted that, much like mathematical algorithms, mathematical abhorithms are, unfortunately, epidemic in many mathematics classrooms. Far from being harmless, they represent a tremendous blind spot in the teaching and learning of mathematics. After putting the finishing touches on the article, I thought I was done writing about abhorithms. Nope.

Change is the Only Constant

I've noticed in the last little while that the conversations I have about the teaching and learning of mathematics have changed. I'm not talking about conversations with colleagues—that is, other math educators. And, I'm not talking about conversations with those who have a vested interest in the teaching and learning of mathematics—that is, math teachers, mathematicians and their ilk. I'm talking about conversations about the teaching and learning of mathematics with members of the general public. Let's look at a few examples that help identify the change that I'm referring to.

As everyone knows, what one does for a living often comes up as a topic of conversation. The title for my job, technically, is "professor of mathematics education;" however, as I've found out over and over again, this title is rather confusing. These days, to avoid confusion, I tell people that I meet that I teach future math teachers. I'll be honest: It's not a phrase that

[1] Chernoff, E. J. (2017). Abhorrent mathematical algorithms: Mathematical abhorithms. *The Variable, 2*(5), 44-50.

I'm completely satisfied with, and I've gone through various versions of the phrase, including "I teach prospective math teachers," "I work with future math teachers," and "I teach classes that future math teachers take while in school," but for whatever reason, "teaching future math teachers" is the one that lands for everybody and allows us to move on in the conversation. Perhaps you know what's coming next.

Once it has been established that I teach future math teachers, recent changes to the teaching and learning of mathematics (e.g., curricular changes, wordy textbooks, calculator use, "new math," etc.) are on the tip of everybody's tongue. Previously, I would often bear the brunt of the ire resulting from peoples' attempts to help children, to no avail, with their math homework. Whether it was parents helping their kids, grandparents helping their grandkids, aunts and uncles helping nieces and nephews, or any other combination of an adult attempting to help a child with their school mathematics homework, the question directed at me was often some version of "Why is math taught differently now?" This question, as I would come to find out through further conversation, was often a veiled admittance that the adult was not able to help a child with their math homework, for whatever reason. And, it would often be followed up with, "Back in my day…" For example, "Back in my day, you would just put a tick mark in the next column and you got on with it. And, for the life of me, I can't figure out why anybody would teach addition from left to right." Or, my favourite, a version of a classic saying: "Back in my day, you didn't understand math, you just got used to it." As I said earlier, though, these conversations are changing.

The Hockey Rink Dressing Room

I realized that this change was taking place during the conversations about the teaching and learning of mathematics that consistently came up for discussion in the hockey rink dressing rooms that I frequent three to four times a week. (I should admit that I use the hockey rink dressing room as a barometer for many things in life. Political, financial, vehicular—you name it, you'll learn a lot if you listen closely when getting dressed and undressed before and after a hockey game.) What came as quite a surprise to me was that the conversations I was used to having about changes in the math class were no longer taking place.

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The exact details aren't necessary; let's just say that I started skating for an additional team. As I got to know the team and the team got to know me, we got comfortable with each other quickly over a few short weeks. Then, it happened: "Hey Chernoff, what's your day job?" To which I gave my now-standard reply about working with future math teachers. "You do, huh. Hey, you know that new math that they're doing in schools…" I thought to myself, "Oh boy, here we go." My new teammate then continued, "I'm the one who does math homework with my daughter." I said something stupid, like, "That's cool." He continued, "Yeah, she's been showing me all these different approaches for adding and subtracting fractions. I didn't really get what she was doing at first because she was using sticks and blocks and drawing pictures. Me, I used to just multiply three times and get the answer." As I began to let out a little smile, I just had to ask: "And, what do you think?"

In the past, this prompt was typically met with derision, but, like I said, things have changed. "Well, math is a tricky subject. I guess it's important to show kids different ways of looking at things if they don't really get it. Hell, the only reason I'm the one who does math homework with our daughter is because I got better grades in math than my wife. I mean, just barely better grades, but she uses it as an excuse to not have to be the math-homework-person in the house." He continued, "You know what though, I'm learning things. Heck, my daughter's even teaching me things about math that would've helped me when I was learning it myself!" It was at this point that I was unable to fully control the smile on my face. I won't go into all the other similar conversations that I've been having, because I'm able to say, with confidence, that the larger message is similar to the one embedded in the dressing room exchange I've just presented. Changes to the teaching and learning of mathematics are taking root, and young people might just be the linchpin to larger acceptance of these changes, as is the case with many other issues (e.g., climate change) in today's world.

Change, as they say, is good. We're all good then, right? Well, if I'm being honest, I'm a little concerned about what's coming next in math class. These next phases are quite crucial for any sea-change in the teaching and learning of mathematics. If we accept, as a premise, a new math class zeitgeist—a math class that embraces mathematics as the science of pattern and order, emphasizes discussion, emphasizes student understanding, and teaches concepts using different strategies according to students' different learning needs; a math class that looks hardly anything like the math class my teammate took all those year ago—fine. There are, however, new responsibilities.

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Abhorithmic Exchanges

Perhaps the most pressing ramification of this new mathematics education world order is that the job of a math teacher just became much more difficult. Yes, the job of math teacher was already difficult; however, in this new world order, the days of "Put your hand down, I'm not taking questions for the remainder of the period" is also over. Be careful what you wish for, as they say.

To help paint this picture, I'm going to recreate a few of my more memorable *abhorithmic* exchanges. These exchanges are ones where I was either (1) the student, (2) the teacher or (3) I overheard while listening to two people discuss school mathematics. As you'll see, this new math classroom, the one that we are perhaps moving towards, may not be any better than the one we are leaving behind.

Multiplication of integers

Easily my most memorable abhorithm came from my Grade 5 teacher, who was teaching the class why a negative number times a negative number resulted in a positive number. As they stood at the front of the room, index fingers pointed at each other at about eye level, they started to slowly move the fingers toward each other, with one of the index fingers changing from a horizontal position to a vertical position, finally resulting in a plus sign. For good measure, the sound "Bwoooooop!" was made during the process. Super memorable, easy to understand and, to be honest, I never got a question involving integer multiplication wrong after that lesson. There's just one problem: "Bwoooooop!" is an abhorithm. The days of abhorithms, arguably, are over. So what if "Bwoooooop!" was replaced with the following exchange?

Egan: Excuse me, Mr. Chernoff, I have a question.

Mr. Chernoff: Yes, Egan… I always love answering your insightful questions.

Egan: Why is it that negative five times negative two is positive ten?

Mr. Chernoff: Well, Egan, remember when I told you that a negative number times a negative number results in a positive number?

Egan: Yes, Mr. Chernoff, I have that written down in my notes from today's class.

Mr. Chernoff: Ok then, is negative five negative?

Egan: Yes.

Mr. Chernoff: And, is negative two negative?

Egan: Yes.

Mr. Chernoff: Well then, the answer is positive *because* we are multiplying two negatives.

That's much better… right?! Before I weigh in, let's consider another example.

Solving Equations
One of my other favourite abhorithms involves solving linear equations. Like most abhorithms, it's simple, concise and, if you just go along with it, you'll never solve a linear equation incorrectly for the rest of time. When solving equations, when you drag a number across the equals sign you change plus to minus or minus to plus. You may have even heard the next line, which goes something like "reasons for this we will not discuss," which applies to many abhorithms. I was even once told that when you drag a number across the equals sign that magic pixie dust falls from the sky and changes the sign. Should you need a visual, $x \mathbf{+ 2} = 4$ becomes $x = 4 \mathbf{- 2}$ because, well, it was dragged across the equals sign which cued the pixie dust sprinkle. Clearly, a mathematical abhorithm is at play. However, as we've discussed, we're at the dawn of a new world order in math class. And so, instead, the following exchange might take place:

Egan: Excuse me, Mr. Chernoff, I have a question.

Mr. Chernoff: Yes, Egan.

Egan: I'm trying to solve this equation, and I'm trying my best to follow the notes you gave us, but why does this minus 13 become a plus 13?

Mr. Chernoff: Remember what I said during the lecture, Egan: positive numbers become negative numbers when you drag them across the equal sign. And, what else did I say during the lecture?

Egan: I think I have that written down, hold on a sec… is this it: positive numbers become negative numbers when you drag them across the equal sign AND negative numbers become positive numbers when you drag them across the equal sign.

Mr. Chernoff: Right. It's nice to see you finally taking notes, Egan. Ok, back to your question. Is the number you're talking about here a negative number?

Egan: Yes. Minus 13.

Mr. Chernoff: And, are you dragging it across the equal sign?

Egan: Yes.

Mr. Chernoff: So…

Egan: …it becomes positive 13?!

Mr. Chernoff: Good! Now, and here's the big question, why?

Egan: Umm, because negative numbers become positive numbers when you drag them across the equals sign…

Mr. Chernoff: Excellent, it seems like you're finally starting to understand things in math class. Good for you!

Much, much better, right?! Look, I know it's obvious that I've been attempting to set you up with these last two examples. Let's dig into them a bit further.

Semicircular Reasoning

So, let's just get to the problem, a new problem in this new math class. The change—that is, the move from *abhorithm* to *abhorithmic exchange*—might appear to be better, at first. After all, students aren't just being told what to do in these instances. They *seem* to be getting explanations, a peek behind the curtains if you will, of the mathematics behind why exactly a negative times a negative is a positive, or what is actually happening when you solve an equation. Alas, an explanation is not what they are getting. And abhorithmic exchanges, I contend, are no better than abhorithms. Even though such exchanges appear to be an improvement over abhorithms, they are nothing but instances of logically fallacious, circular reasoning.

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As you probably know, circular reasoning is a logical fallacy where propositions are supported by premises, which in turn are supported by the same propositions—thus, creating a circle. Perhaps some non-math-class examples are appropriate at this point. *Nineteen-year-olds have the right to drink because it's legal for them to drink* is an example of circular reasoning (and, further, an example of begging the question). Here's another: *Something can't come from nothing; thus, the Big Bang cannot have happened*. The issue, of course, is that the conclusion is assumed in the premise. Believe it or not, I have a favourite example because, well, it really hit home for me.

Consider the conversations many teenagers have about curfew with their parents. You've probably had these conversations yourself. Maybe, when you were younger, you had a curfew at, say, 11:00 p.m. At some point, you probably began to question your curfew. You may have even asked your Mom, your Dad, or whomever about the details of your curfew. Try as you might, however, you were unable to crack the reasoning associated with your curfew. Chances are, this is because the conversation about your curfew involved circular reasoning. You may have asked why you have to be home by 11:00 p.m., to which they replied that 11:00 p.m. is your curfew. You may have even appealed to the fact that all of your friends were able to come home at a later time than you, say 12:00 a.m., and you would ask why you, too, weren't allowed to stay out to 12am, to which they probably replied again, to your great frustration, that your curfew was at 11:00 p.m. Here again, the proposition (you must come home at 11:00 p.m.) is supported by the premise (curfew is at 11pm), which is supported by the proposition (you must come home at 11:00 p.m.). Because of this, there is a "circle" in the reasoning, meaning that, in essence, your curfew conversation is going nowhere. In general, there is a logical form to circular reasoning: *X is true because of Y. Y is true because of X.* This brings me to my use of the term "semicircular reasoning" as opposed to "circular reasoning" in the title of and throughout this article.

My use of "semicircular reasoning" is an attempt to not lose the forest for the trees when looking at this potential issue in the math class. In other words, I want to avoid the argument that, if the reasoning does not explicitly follow the logical form of circular reasoning then, somehow, we're all off the hook. Semicircular reasoning, then, is a term I use to describe any instance of an abhorithmic exchange that even has a hint of circular reasoning. Looking back to the first example I gave in the math class, when the teacher is asking the student whether or not the number they see in front of them is negative or not, they are doing so with the conclusion being assumed in the premise. This particular abhorithmic exchange, then, is an example of semicircular reasoning. And at this point, we need to ask ourselves if sharing semicircular reasoning with our students is any better than "Bwoooooop!" Similarly, asking the student whether or not they dragged the number across the equal sign does not get us any closer to a mathematically sound justification and explanation as to what is really going on when we're solving linear equations in the math classroom.

At this point, we need to ask ourselves if sharing semicircular reasoning with our students is any better than "Bwoooooop!"

Analyzing Abhorithmic Exchanges for Semicircular Reasoning

Concerningly, perhaps alarmingly, if you really start looking and listening for the use of semicircular reasoning in the math class then it might start to appear more than you'd like. Consider the following example.

> *Egan:* Excuse me, Mr. Chernoff, I have a question.
>
> *Mr. Chernoff:* Of course you do, Egan.
>
> *Egan:* I'm trying my best to follow the notes that I wrote down during your lecture… I was just wondering, why do we change the division sign to a multiplication sign and put the number on the bottom on the top and the number on the top on the bottom for the second number?
>
> *Mr. Chernoff:* Well, Egan, what question are you working on?
>
> *Egan:* This one, from the homework: $3 \div \frac{1}{2}$.
>
> *Mr. Chernoff:* Hmm, maybe you weren't listening during the lecture, Egan, but let's see if you maybe remember. What did I say was the first rule for dividing fractions?
>
> *Egan:* You never divide fractions?
>
> *Mr. Chernoff:* Right, we never divide fractions. Instead, we…
>
> *Egan:* Multiply…
>
> *Mr. Chernoff:* Right. And how do we multiply fractions?
>
> *Egan:* We change the division sign to a multiplication sign and then put the number on top on bottom and the number on bottom on top.
>
> *Mr. Chernoff:* See, you understood what you were doing all along.

Analyzing the abhoritimic exchange through the lens of semicircular reasoning, the student is no closer to learning the mathematical underpinnings as to why you change the sign and put the number on top on the bottom and the number on the bottom on top. The conclusion—that is, to change the sign and put the number on the top on bottom and the number on the bottom on top, is assumed in the premise. What we have then, are instances of students looking for the conclusion in their textbooks and elsewhere and then being rewarded when they are able to identify a division of fractions question. Unfortunately, semicircular reasoning is definitely not the change that is supposed to be taking place in the math class. Consider the following exchange:

> *Egan:* Excuse me, Mr. Chernoff, I have a question.
>
> *Mr. Chernoff:* Sure thing, Egan.
>
> *Egan:* You said that when we're converting from a decimal to a percent that we move the decimal point two places to the right.
>
> *Mr. Chernoff:* That's exactly what I said.
>
> *Egan:* I guess my question is why we're moving it to places to the right?
>
> *Mr. Chernoff:* Because you're multiplying by 100, Egan!
>
> *Egan:* Ok, I guess…

The above exchange reminds me, and perhaps it reminds you, of the curfew conversation. Circular reasoning should not have worked for your parents when you asked to stay out a bit later; alas, it did. Semicircular reasoning does not and should not work in math class. Period.

Moving the Goalposts

The argument could be made that this potential issue could be avoided by teaching the abhorithms and "just getting on with it." After all, if students learn to simply move the decimal over when converting a percentage to a fraction, then the damage, I contend, is localized. However, if they listen to the fallacious reasoning that is being used to support the abhorithm, then there are now two instances of damage. In the worst-case scenario, a student, let's say it's a bright student, notices a pattern that's starting to emerge while they're learning mathematics—that is, whether the teacher is teaching multiplication of integers, solving equations, fraction division, or converting between decimals and percentages, the reasoning behind all of these topics appears to the student to be one and the same, so maybe math isn't that difficult after all. In other words, the damage has spread.

As Garth Algar famously said, "We fear change." When it comes to a math class that is purported to be digging deeper into the mathematics but, instead, is relying on semicircular, fallacious reasoning when attempting to clarify the school mathematics that students are attempting to learn, just give me the abhorithm (something I never, ever thought I'd say), because it might be less damaging in the end. Coincidentally, some might say ironically, utilizing the logical fallacy of semicircular reasoning to explain mathematical abhorithms could even result in another logical fallacy; me, I'd rather just keep the goal posts where they are. After all, semicircular reasoning in not a teaching strategy, because it's not.

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Ilona & Nat,
Editors

Saskatchewan Mathematics Teachers' Society