
*Alternate Angles is a column on problems from multiple perspectives: various methods that could be used to solve them, insights we get from their solution, the new paths that they can lead us to once they have been solved, and how they can be used in the classroom.*

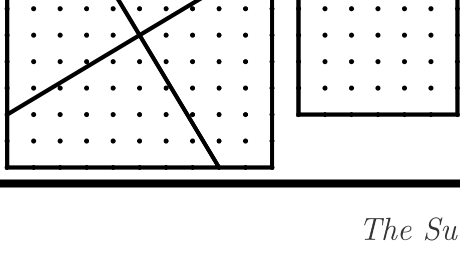
## #13. Slicing Squares

*Shawn Godin*

WELCOME back problem solvers! This is the first article of the continuation of my column from *The Variable*. I hope you enjoy the column. I plan to continue it for years to come!

Sometimes the most innocent questions have something lurking under the surface. I recently surveyed the last 60 years worth of the *Ontario Mathematics Gazette* for a column I do in that publication, *What's the Problem?*, similar in nature to *Alternate Angles*. I found this problem in the *Abacus* section. The *Abacus* is now the part of the *Gazette* dedicated to elementary teachers, but in the early years, it was just a collection of activities at all levels. The problem below appeared in what I originally thought was the very first issue (Abacus 25(1), p. 10).

> *Cut out the ten-centimetre and the six-centimetre square shown below. Working clockwise, we mark a point on each side of the ten-centimetre square two centimetres from the corner. We then join the points on opposite sides. Now cut along the lines and reassemble the four pieces and the six-centimetre square to make one large square.*



At first glance, it isn't much of a "problem". Kids could cut out the shapes, use a little spatial sense to build a square, and be done with it. The problem was likely designed to have students internalize the properties of a square in an abstract way. When a problem has been solved, it is always worthwhile to ask oneself "Are we done, or can we squeeze anything more interesting out of this problem?" Let's find out.

As problem solvers, we should be used to going through George Pólya's [1] four steps of problem-solving: understand the problem, devise a plan, carry out the plan, and look back. The final step, looking back, is arguably the most important. At that step we consider:

- is our result reasonable?
- is our method sound?
- is there another way we could have obtained our result?
- can we use our result or method in other problems?

The last question is particularly important for mathematicians. It leads to connections between different areas of mathematics, as well as generalization. Coming back to our own problem, once the pieces have been rearranged to make the larger square, do any new questions come to mind? Here are a few that came up for me:

1. Is this the only way to cut up and rearrange the squares into a large square?
2. Is there something special about *these* squares, or could we do the same thing with other squares?
3. Is there something else lurking in the shadows?

By my third question, I am wondering if there is some general principal at work.

As you investigate your questions you may find that some of them are tied together, or you may find that other questions arise. Some questions, you may realize, you cannot answer, and that's OK. The more you do this process, the better you will get at it. This will help your problem-solving skills in general and help you develop stronger connections between mathematical concepts.

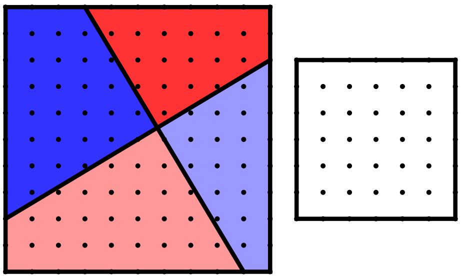


Figure 1: *Moving the cut*

To investigate the first problem, we might start by moving one of our cuts. Let's move the cut from the top to the bottom over one unit to the right, so that we have something similar to Figure 1. Rearranging the pieces will eventually lead you to Figure 2.

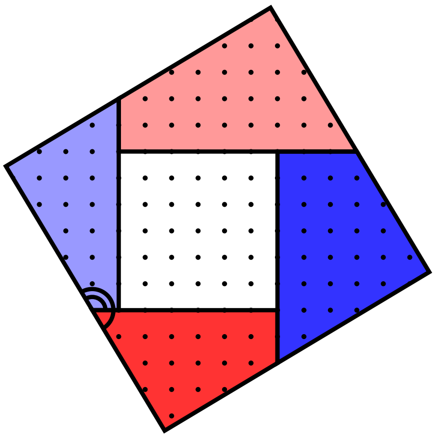


Figure 2: *Assembling the square*

If you examine the sides of the big square in Figure 2, you will notice that the two pieces that make up each side were together in the Figure 1, albeit in a different orientation. Hence, the length of the sides of the square is equal in length to that of the angled cuts we made to the original square on the left. Considering the two indicated angles in Figure 2, when located in Figure 1, they were interior angles so they are indeed supplementary. This may lead us to the following:

> **Conjecture:** *As long as we draw two perpendicular cuts though our big square, it can be rearranged into a "square around a square."*

I created a *GeoGebra* sketch to investigate this conjecture, shown in Figure 3. The three points allow you to change the location and slope of the cuts, while keeping the cuts perpendicular. When the animate button is pressed, the four pieces move to the side and rearrange themselves in the way we did for the original problem. I suggest you take some time to play with the sketch and see what you think.

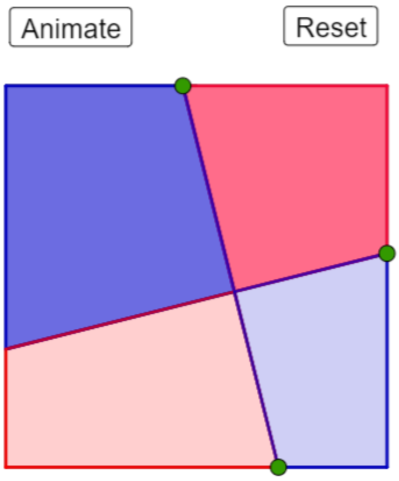



Figure 3: *Screenshot of* GeoGebra *sketch*

Hopefully, by playing with the *GeoGebra* sketch you noticed that our cut square will always rearrange into a square with a square hole, which could be filled with a second square. However, the sizes of the new square changes, depending on the slopes of the cuts. This is probably not much of a surprise. We had noticed earlier that the length of the cut is equal to the side length of the new square. Where does the second square, the hole in the sketch, come in?

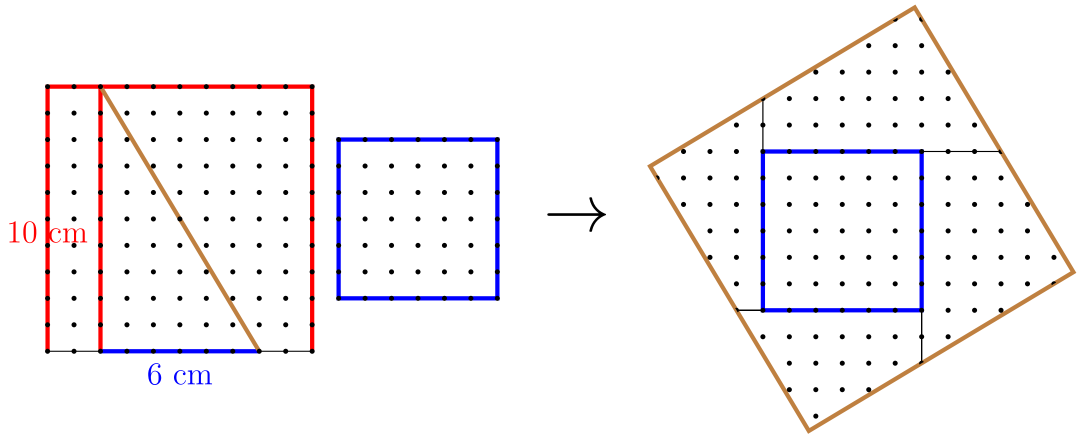



Figure 4: *How the squares are related*

Figure 4 shows that the side length of the other square shows up in determining our cut. The third of our questions, "*is there something else lurking in the shadows?*", is now answered. The harmless little question that we originally looked at was really a statement of the Pythagorean theorem in disguise! Figure 5 shows a screenshot of another *GeoGebra* sketch showing the Pythagorean theorem. In this sketch, you can move the location of the two cuts on the larger square or change the size of the smaller square by moving the indicated points. To see how I made the sketch, you can see a video here.

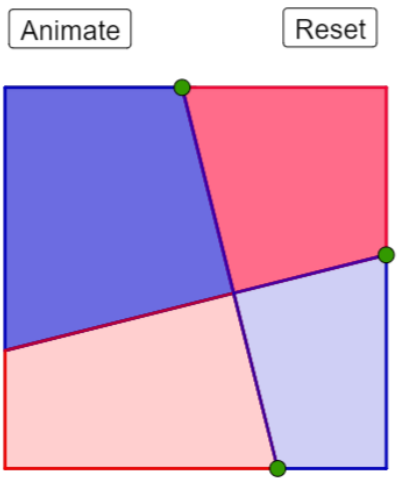



Figure 5: *Screenshot of* GeoGebra *sketch*

I hope you enjoyed the Pythagorean surprise! Remember, when solving problems, take some time at the end to think of your own questions and explore them as far as they will take you. This is a great habit to have and a great one to instill in our students. Until next time, happy problem solving.

## References

[1] Pólya, George, *How to Solve it, second edition*, Princeton University Press, 1957.

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The problems below were also featured in *The Abacus* that appeared in *Ontario Mathematics Gazette* 25(1), edited by Trevor Brown.

**1.** *In bell ringing with three bells, there are six different orders in which they can be rung for example,* $(1, 2, 3)$, $(2, 3, 1)$, *etc. The aim is to produce all six orders before repeating one of them. No bell is allowed to move up or down more than one place from one order to the next (so* $(1, 2, 3)$ *cannot be followed by* $(2, 3, 1)$*). Investigate. (page 4)*

**2.** *There is something very special about 6174. Select any four-digit number whose digits are not all equal and arrange the digits to form the greatest number possible, that is, put the digits in decreasing order. Then form the reverse and subtract the two numbers.*
*Continue this process of forming the greatest possible number from the difference and subtracting its reverse. The original number in the example below is 4218. The process ended in five steps with 6174.*
*Can you find a four-digit number whose digits are not all equal for which this process requires more than seven steps to produce 6174? (page 11)*

**3.** *Consider the transformation.*

$$\frac{a}{b} \to \frac{a+2b}{a+b} \quad \text{where } a,\ b \text{ are whole numbers.}$$

*example:*

$$\frac{1}{1} \to \frac{3}{2} \to \frac{7}{5} \to \frac{17}{12} \to \dots$$

*Investigate transformations of this kind. (page 12)*