Alternate Angles is a column on problems from multiple perspectives: various methods that could be used to solve them, insights we get from their solution, the new paths that they can lead us to once they have been solved, and how they can be used in the classroom.

#14. A Tale of Two Cities

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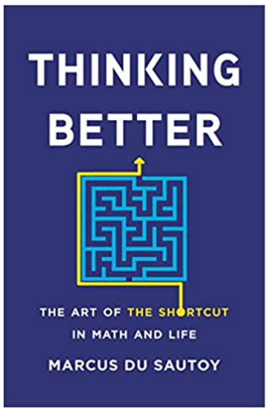
WELCOME back problem solvers. As an avid problem solver myself, I'm always on the lookout for good problems. I often go on active searches: scouring math contests, journal articles, YouTube videos, and books on my bookshelf. Sometimes, problems present themselves to me in something I'm watching or reading. Other times, though, problems are inspired by something unexpected[1]. I usually try to make a mental note of these discoveries, and in some cases, I actually revisit them.



I recently read *Thinking Better : The Art of the Shortcut in Math and Life* by Marcus Du Sautoy, the Simonyi Professor for the Public Understanding of Science at the University of Oxford. The book is broken up into 10 chapters and 9 "pit stops". The chapters open with a puzzle whose solution is related to the specific shortcut being discussed. Through problems, history, and great story telling, Du Sautoy leads the reader through short cuts like patterns, diagrams and probability, and spices up the story with some real-world connections during pit stops such as memory, art and neuroscience. Overall a great read that I would recommend to ***Alternate Angles*** readers.

[1] For instance, I got the idea for what became question 4 on the 2011 Fryer Contest digging through my change collection while in the drive-through line at Tim Hortons.

Chapter 4, *The Geometric Shortcut*, opens with the following problem which I've adapted for the Saskatchewan context:

> *There are ten people in Regina and five in Saskatoon. The distance between the two cities is about 260 km. Where should they meet so that the total distance travelled by all fifteen people is the smallest?*

This is a nice problem to give your students and ask for gut reactions. Give them a short time to think about it, then give them the following choices:

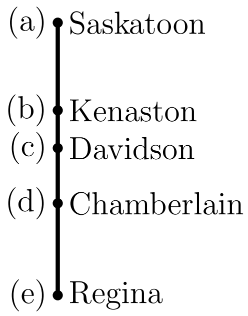
(a) Saskatoon

(b) Kenaston

(c) Davidson

(d) Chamberlain

(e) Regina



A discussion of why they made their particular choices would be fruitful. In some cases their intuition may be bang on. However, there may be more to be learned from their incorrect guesses. As the problem gets explored, it might help them fix some of their misconceptions. In other cases, they may be wrong, but their intuition may be picking up something that is important.

Going into this process, it is important to anticipate some of the students' responses. I suspect many will answer "Davidson", as it is near the midpoint between the two cities. Some others may choose Chamberlain as it is about $\frac{1}{3}$ of the way from Regina, and $\frac{2}{3}$ of all the people are starting in Regina. It would be interesting to see if students could determine which choice between these two would be best.

As is often the case, considering an easier problem may help us understand the problem at hand. Let's consider three people: two in city A and one in city B (maybe in our original problem, the people are being environmentally conscious and car-pooling, five people to a car), with the cities 6 units apart (so that we can easily calculate the midpoint, as well as the $\frac{1}{3}$ and $\frac{2}{3}$ points). We can even model this with manipulatives, if you wish. In Figure 1, we can see that if the meeting place is at the midpoint, it requires a total of 9 units of travel, and yet if we are $\frac{1}{3}$ of the way from the city with more people, we only require 8 units of travel. It seems we have our answer!

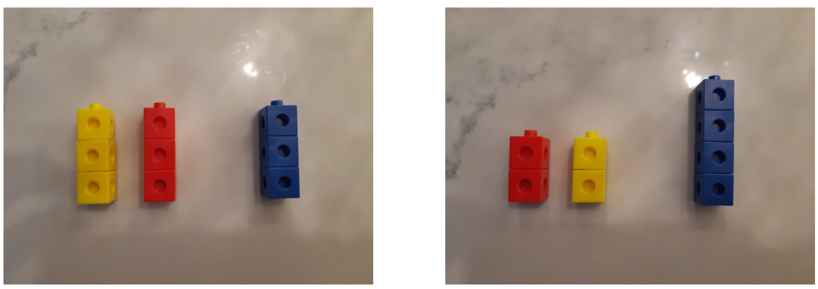


Figure 1: *Modelling a simplified version*

However, if we look at more possibilities for the simplified problem, we see something else happening.

Distance from A	**Total distance**
0	6
1	7
2	8
3	9
4	10
5	11
6	12

It seems that the best place to hold the meeting is in city A. Not only that, we seem to have uncovered a linear relationship in the process. Let's see if we can verify this with a little algebra. Let x represent the distance, in units, that the meeting is held from city A and y represent the total distance travelled by all people to get to the meeting. By definition, each of the people in city A have to travel x units, while the person in city B have to travel $6 - x$ units. Thus

$$\begin{aligned} y &= 2x + (6 - x) \\ y &= x + 6 \end{aligned}$$

which verifies the linear relationship. Since we want to minimize the total distance, and the relationship is increasing, we want to make x as small as possible, that is, 0.

If we return to our original problem and let x represent the distance, in kilometres, that the meeting is held from Regina and y represent the total distance travelled by all people to get to the meeting, we get

$$\begin{aligned} y &= 10x + 5(260 - x) \\ y &= 5x + 1300. \end{aligned}$$

So we can conclude that the best place to hold the meeting is in Regina, if we want to minimize the total distance travelled.

Can we justify our result in another way? Returning to Figure 1, we see that, for the simplified problem, making the person in city B travel 1 unit more corresponded to *each* of the people in city A traveling 1 unit less, resulting in the total distance being 1 unit less. Thus, however far the people from A have to travel, if we reduce their travel, we reduce the total. Hence, the total distance will be minimized if we minimize the distance the people from A have to travel, which occurs when everyone meets at A. For the main problem, for ever kilometer that we are closer to Regina, the people from Saskatoon have to travel a total of 5 kilometres more, while the total for the people from Regina is reduced by 10 kilometres, leading to 5 kilometres fewer in total. Thus we should minimize the distance travelled by the people from Regina by holding the meeting in Regina. It would seem at this point we can generalize to say that if we have a people from city A and b people from city B, then to minimize the total distance travelled for the $a + b$ people you should meet in city A if $a > b$ and in city B if $b > a$ (what happens if $a = b$?).

So we have not only solved the problem, but solved a generalized version as well. Is there anything left to do? What if instead of people from two cities we had people from three? Consider the following:

> *There are ten people in city A, five in city B and two in city C. The distance between city A and B is 100 km as is the distance from B to C. The cities lie on a straight line. Where should they meet so that the total distance travelled by all seventeen people is the smallest?*

We can use a method similar to our first problem and look at two cases:

Case 1: We meet somewhere between cities A and B.

If we let x represent the distance the meeting is from city A and y represent the total distance, we get

$$\begin{aligned} y &= 10x + 5(100 - x) + 2(200 - x) \\ y &= 3x + 900 \end{aligned}$$

from which we can conclude that if we are meeting between cities A and B, the best place to meet would be at A for a total distance travelled of 900 km.

Case 2: We meet somewhere between cities B and C.

If we let x represent the distance the meeting is from city B and y represent the total distance, we get

$$\begin{aligned} y &= 10(100 + x) + 5x + 2(100 - x) \\ y &= 13x + 1200 \end{aligned}$$

from which we can conclude that if we are meeting between cities B and C, the best place to meet would be at B for a total distance travelled of 1200 km.

Therefore, the best place to meet would be city A.

We may conjecture that, like the first problem, the best place to meet would be the place with the most people. If we change the location of the people, for example to 5 in A, 10 in B and 2 in C, we get the same thing. Always go to the place with the most people. Interested readers should try some, or all, of the remaining permutations of 10, 5, and 2 amongst A, B, and C. Even if you change the distances between the cities, you always end up in the town with the 10 people. For example, if we there are p km between A and B and q km between B and C, with 10 people at A, 5 at B and 2 at C, then if you meet between A and B, the total distance would be

$$y = 3x + 7p + 2q$$

which in minimized if you meet at A (distance is $7p + 2q$). If you meet between B and C, the total distance would be

$$y = 13x + 10p + 2q$$

which is minimized when we meet at B (distance is $10p + 2q$). However, no matter what p and q are,

$$7p + 2q < 10p + 2q,$$

which puts our best choice at A. All the evidence seems to point towards meeting at the city with the most people, as we conjectured, and we can give ourselves a smug pat on the back.

However, what if there were 10, 8 and 5 people rather than 10, 5 and 2? Now, depending on the arrangement of the people, the best choice could be meeting in the town with 10 people, but in some cases it could be meeting in the town with 8, and in others, in the town with 5. How can we figure out what is going on? We could go back and do some algebraic analysis, as we have done so far, but if we expand the problem to more and more cities, this will become more and more cumbersome. We need a different insight.

Let's return to the simplified version of the original problem and to our linking cubes. Another way we can look at things is as in Figure 2. If we pair up the person in city B with a person in city A, we see that no matter where we have the meeting, the total distance travelled by the pair will be 6 units. Thus our total for everyone is 6 units plus however far the other person from city A has to travel.

Figure 2: *A different point of view*

Let's use this insight for the case where 10 people are from city A, 8 are from B, and 5 are from C. Start by pairing the 5 people from C with 5 from A. No matter where we have the meeting, the total distance that these 5 pairs of people will travel will be 5 times the distance from A to C. Now, there are 5 "unpaired" people from A who we will pair with 5 people from city B. If the meeting is between A and B, these 5 pairs will have to travel a total distance of 5 times the distance from A to B. If we meet between B and C, they will have to travel even further. That leaves 3 "unpaired" people from city B. Hence, if we meet in city B, the total distance will be minimized.

I created a graph in *Desmos* to model this situation, shown in Figure 3. In the model, cities A and C are fixed at 0 and 10, respectively. City B, as well as the meeting place, M, can be dragged to different positions between A and C. The total distance travelled is indicated by d (80.36 units in Figure 3). Above each city is the number of points corresponding to the number of people in the city. The number of people in cities A, B, and C are a, b, and c, respectively, and can adjusted, via the sliders, to any value from 1 to 20. You can see how people from A and C can pair so the total distance between the two of them is the distance between A and C, regardless of the position of M. Similarly, the people from city B get paired up with any "leftovers" from either city A or C. The segments are shifted a bit for city B in case M is on the "wrong" side so that the paired segments would not overlap. You can play with the model to get a feel for our second insight.

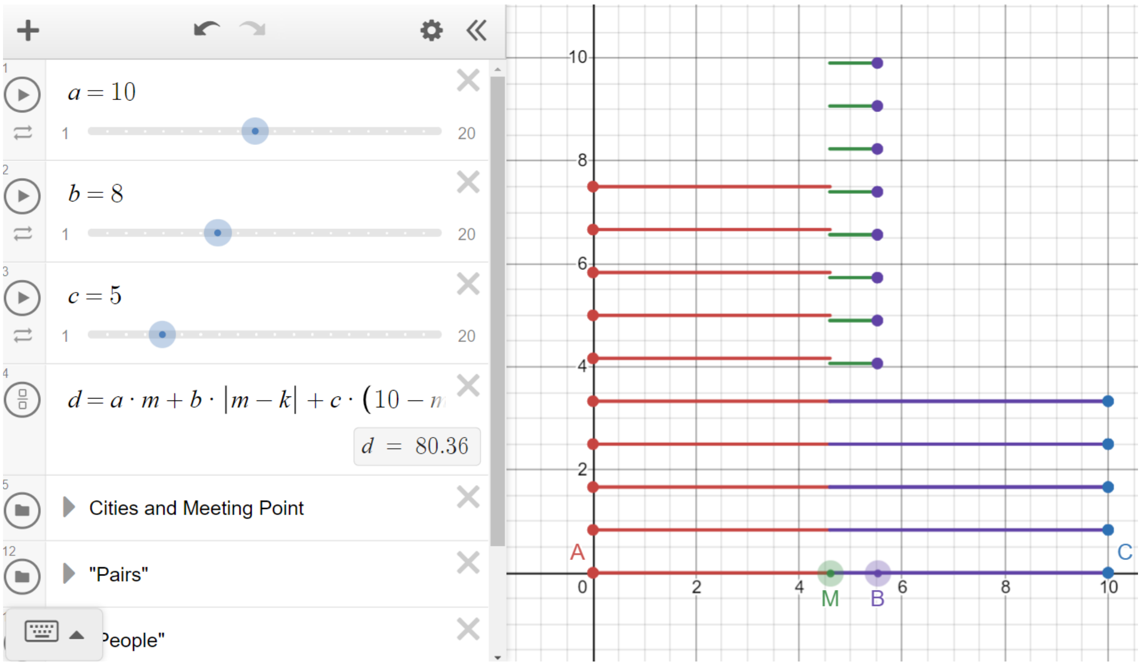


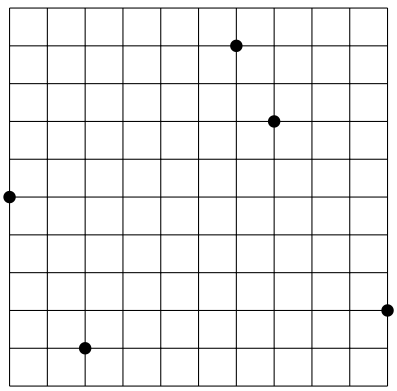
Figure 3: *A Model for Three Cities*

The three folders at the bottom left of Figure 3 contain some of the computations needed to generate the model. I discuss the creation of this graph in a video. Interested readers are encouraged to play with the model and check out the video.

Let's conclude with the general case, where we have groups of people on a straight line and we want to minimize the distance that they all have to travel. Start with the people that are furthest apart and pair them, then the next furthest pair, and so on. If you are left with one unpaired person, meet at their town. If there are two closest people (in the middle), you can meet anywhere in between their towns. For example, suppose you had 6, 3, 9, and 7 people at A, B, C, and D, respectively. Since we have 25 people, we should meet at the town of the middle person, person 13, which would be town C, regardless of how far apart the towns are separated. Similarly, if we had 58, 62, 39, 120, and 39 people at A, B, C, D, and E, then since there are 318 people, we should go between person 159 (city C) and person 160 (city D). Thus, meeting in city C or city D or any place in between would be optimal. The interested reader should try to verify these claims.

The problem can even be generalize to two or three (or more!) dimensions. For example, consider the following:

> *Five points are placed on a grid. If you can only move parallel to the axes, which point would be the best "meeting point", that minimizes the distance for travelers originating at the five points?*



As a hint, since this is in two dimensions, you can consider it as two separate problems: determining the best x-coordinate to meet and the best y-coordinate to meet.

I hope you enjoyed this problem as much as I did. Remember, it's a good habit of mind for you and your students to play with problems beyond the confines of what was originally asked. Many math problems will lead you to some interesting places if you are willing to do a bit of exploring.

Until next time, happy problem solving!

Shawn Godin is a retired mathematics teacher and department head living in Carleton Place, Ontario, just outside of Ottawa. He strongly believes in the central role of problem solving in the mathematics classroom. He continues to be involved in mathematical activities: leading workshops, writing articles, working on local projects and helping create mathematics contests. Comments and questions are welcome at `GodinMathStuff@gmail.com`.