



Alternate Angles is a column on problems from multiple perspectives: various methods that could be used to solve them, insights we get from their solution, the new paths that they can lead us to once they have been solved, and how they can be used in the classroom.



#15. Digging Deeper

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MANY problems allow for deeper exploration if the solver is willing to ask questions and then attempt to answer them. This is a process that should be modelled to students to get them into the habit of thinking beyond the confines of the problem at hand. Asking ourselves questions is a great habit of mind to cultivate, whether we are solving a math problem, watching the news, or reading an article. In this issue, we will start with Problem 2 from Section B of the [2021 Canadian Intermediate Mathematics Contest](#) hosted by the [The Centre for Education in Mathematics and Computing \(CEMC\)](#) at the University of Waterloo.

(a) In Figure 1, trapezoid $ABED$ is formed by joining $\triangle BCE$ (which is right-angled at C) to rectangle $ABCD$. If $AB = 3$, $CE = 6$ and the area of trapezoid $ABED$ is 48, determine the length of BE .

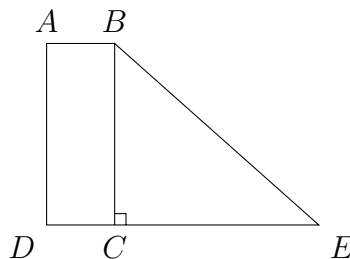


Figure 1

(b) In Figure 2, circles with centres P and Q touch a line ℓ (that is, are tangent to ℓ) at points S and T , respectively, and touch each other at point X . Because the circles touch at X , line segment PQ passes through X and so $PQ = PX + XQ$. Because the circles touch ℓ at S and T , PS and QT are perpendicular to ℓ . If the circle with centre P has a radius of 25 and the circle with centre Q has a radius of 16, determine the area of trapezoid $PQTS$.

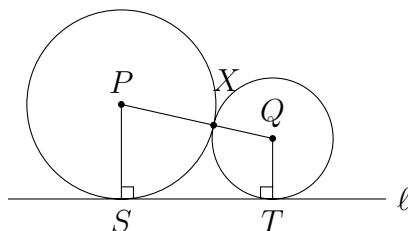


Figure 2

(c) Figure 3 is formed by adding a circle of radius r to Figure 2 that touches ℓ and each of the two other circles, as shown. Determine the value of r .

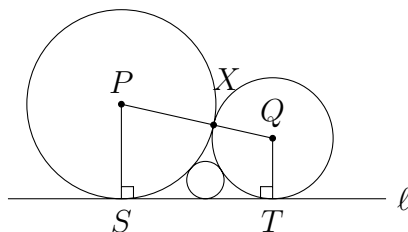


Figure 3

We can see from the problem set-up that parts (a) and (b) are setting the students up to solve part (c). Much of the circle geometry that I learned in school is no longer in the curriculum (in Ontario, at least). As such, the problem gives the following properties to the student:

- The radius to the point of tangency of a tangent is perpendicular to the tangent.
- When two circles are tangent, their centres and the point of tangency are collinear.

I will leave part (a) to the interested reader and dive into part (b). Drawing in segment QY , with Y on PS such that $QY \perp PS$, we create the right triangle PQY , shown in Figure 4.

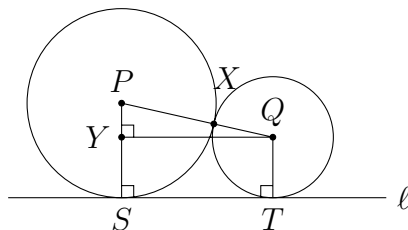


Figure 4

In $\triangle PQY$, $PY = 25 - 16 = 9$ and $PQ = 25 + 16 = 41$. The Pythagorean theorem then yields

$$\begin{aligned} QY^2 &= 41^2 - 9^2 \\ &= 1600 \\ \therefore QY &= 40 \end{aligned}$$

From our set up $QY = TS$, so

$$\begin{aligned} \text{Area of } PQTS &= \frac{1}{2}(25 + 16)(40) \\ &= 820 \end{aligned}$$

At some point during the solution process, hopefully you wondered: “can we generalize this”? In other words:

Given two circles with radii r and R with $r < R$ that are externally tangent. If these circles are both tangent to a line ℓ (not through the point of tangency of the circles) at points S and T , how is the length ST related to r and R ?

Fortunately, our method for solving part (b) can be used here, replacing 16 and 25 with r and R , respectively. Creating point Y like we did in our previous solution, we get the situation in Figure 5:

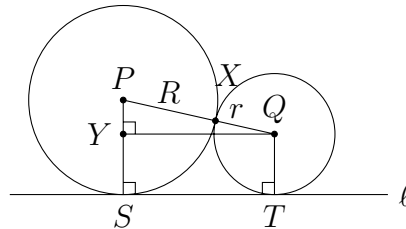


Figure 5

Then, in $\triangle PQY$, $PY = R - r$, and $PQ = R + r$. Once more the Pythagorean theorem yields

$$\begin{aligned} QY^2 &= (R + r)^2 - (R - r)^2 \\ &= (R + r + R - r)(R + r - R + r) \\ &= 4Rr \\ \therefore ST &= QY = 2\sqrt{Rr} \end{aligned}$$

What an interesting result!

The expression \sqrt{Rr} is called the *geometric mean* of the values R and r . In general, if we had quantities $x_1, x_2, x_3, \dots, x_n$, then the better-known *arithmetic mean* is given by

$$AM = \frac{x_1 + x_2 + x_3 + \dots + x_n}{n},$$

while the geometric mean is given by

$$GM = \sqrt[n]{x_1 x_2 x_3 \dots x_n}.$$

The geometric mean shows up in interesting places in mathematics, and the two means satisfy the inequality $GM \leq AM$, with equality happening if and only if all the numbers are the same.

Before we attack part (c) of the problem, let's think about the implication of our generalization. We now know how the distance between the points of tangency of two circles that are externally tangent and tangent to a line ℓ is related to the radii of the two circles. Hence, if we let ℓ be the x -axis and let the centre of one of the circles fall on the y -axis, then the centre of one circle will be $(0, R)$, and it will be tangent to the x -axis at the origin. The second circle will then be tangent at the point $(2\sqrt{Rr}, 0)$ and hence have centre $(2\sqrt{Rr}, r)$. So, if we use [Desmos](#), or some other graphing calculator, we can easily set up a simulation like the one pictured in Figure 6. You can access my original graph [here](#).

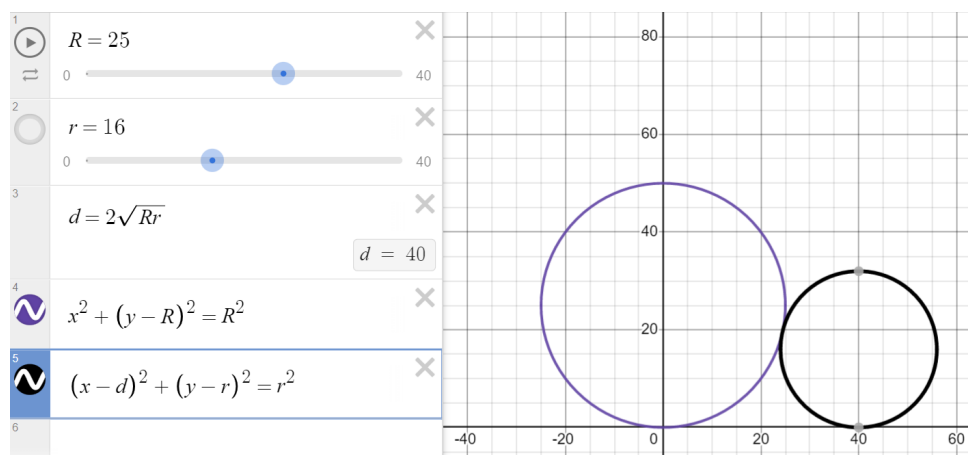


Figure 6: A simulation in *Desmos*

Now on to part (c). Rather than solving the original problem, let's jump straight to the generalization. That is:

Two circles, with radii r_1 and r_2 , are externally tangent and tangent to a line ℓ , not passing through the point of tangency of the circles. A third circle, with radius r , is drawn so that it is tangent to ℓ and externally tangent to the other two circles. How does r depend on r_1 and r_2 ?

Adding the third circle our diagram becomes like the one in Figure 7.

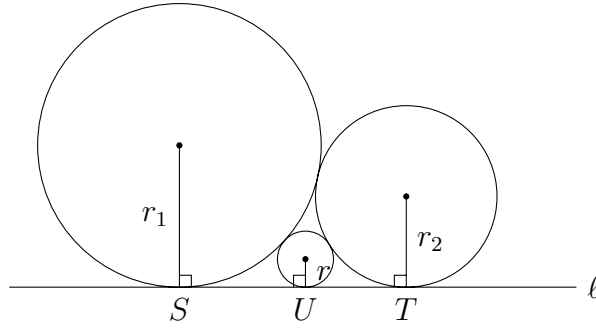


Figure 7

From our first generalization, we know the distance from S to U will be $2\sqrt{r_1r}$, the distance from U to T will be $2\sqrt{r_2r}$, and the distance from S to T is $2\sqrt{r_1r_2}$. As U is on line segment ST we get

$$\begin{aligned} SU + UT &= ST \\ 2\sqrt{r_1r} + 2\sqrt{r_2r} &= 2\sqrt{r_1r_2} \end{aligned}$$

which we can simplify and solve for r as

$$\begin{aligned} \sqrt{r_1r} + \sqrt{r_2r} &= \sqrt{r_1r_2} \\ \sqrt{r}(\sqrt{r_1} + \sqrt{r_2}) &= \sqrt{r_1r_2} \\ \sqrt{r} &= \frac{\sqrt{r_1r_2}}{\sqrt{r_1} + \sqrt{r_2}} \\ \therefore r &= \frac{r_1r_2}{r_1 + 2\sqrt{r_1r_2} + r_2}. \end{aligned}$$

Why stop here? Putting this expression into our expression for the distance between S and U yields

$$SU = 2\sqrt{r_1r} = \frac{2r_1\sqrt{r_2}}{\sqrt{r_1} + \sqrt{r_2}}.$$

Those that are interested can now modify your *Desmos* graph to include the third circle. You can check out my modified version [here](#).

Let's keep digging. Suppose we start with the original two circles having radii of 1 unit each. What will be the radius of the small circle? Running it through our formula yields

$$r = \frac{1 \times 1}{1 + 2\sqrt{1 \times 1} + 1} = \frac{1}{4}.$$

Hence, if we have two circles with radii of 1 externally tangent and tangent to a line, not through their common point of tangency, then a circle of radius $\frac{1}{4}$ can be placed so it is tangent to the two circles and the line.

If we continue the process, we would find the circle between the circles of radii 1 and $\frac{1}{4}$ to have radius $\frac{1}{9}$. Hmm... something seems to be going on here. If we do it one more time, the circle between the circles of radii 1 and $\frac{1}{9}$ has radius $\frac{1}{16}$. Since there are two circles of radius 1, we can draw in two circles each of radii $\frac{1}{9}$ and $\frac{1}{16}$ to get Figure 8.

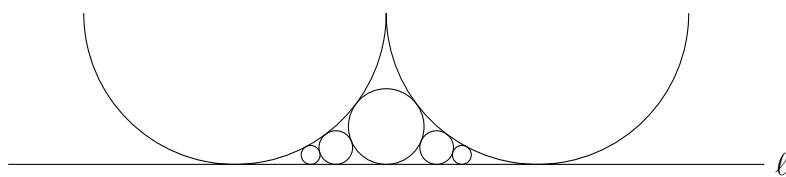


Figure 8 : Tangent circles of radius 1, $\frac{1}{4}$, $\frac{1}{9}$, and $\frac{1}{16}$

It seems that the radii are just the reciprocals of the squares $1 = \frac{1}{1^2}$, $\frac{1}{4} = \frac{1}{2^2}$, $\frac{1}{9} = \frac{1}{3^2}$, and $\frac{1}{16} = \frac{1}{4^2}$. Does this process continue? Suppose we have circles of radii 1 and $\frac{1}{n^2}$, what is the radius of the circle between them? From our earlier result we get

$$\begin{aligned} r &= \frac{1 \times \frac{1}{n^2}}{1 + 2\sqrt{1 \times \frac{1}{n^2} + \frac{1}{n^2}}} \\ &= \frac{\frac{1}{n^2}}{1 + \frac{2}{n} + \frac{1}{n^2}} \times \frac{n^2}{n^2} \\ &= \frac{1}{n^2 + 2n + 1} \\ &= \frac{1}{(n+1)^2} \end{aligned}$$

so the result follows, by induction.

This configuration is closely related to [Ford circles](#). Further investigation may lead you to [Farey sequences](#) and their properties. If you dropped the tangent line and just focused on tangent circles you may end up investigating [Apollonian gaskets](#) or [Steiner chains](#). The limit of our discoveries is only limited to our imagination. Have fun digging deeper into these problems and more!

Readers of *Alternate Angles* may be interested in my YouTube channel, [Mathematical Mish Mash](#). I have created a series of videos to accompany this column, looking at exploring the problem with *GeoGebra*.

- [Part 1](#) shows how to create a sketch in *GeoGebra* that will mimic what we did with *Desmos* in Figure 6 as well as adding the third circle seen in figure 7.
- [Part 2](#) recreates what was done in Part 1, with the “first” circle and tangent line more “free”.
- [Part 3](#) recreates the same sketch, however, geometry is used to define the second circle (without need of a slider) and the third circle (without need of the formula from this article).
- [Part 4](#) shows you how to create some custom tools in *GeoGebra*. The tools are related to our sketches so far and allow you to easily extend them to things like Figure 8.
- [Part 5](#) shows an advanced technique that further allows us to sculpt our *GeoGebra* sketches. In Part 4, the sketch may not behave as we originally intended if we add more circles. The technique allows us to “pick” the specific points of intersection used to create the circles.

I have done a few other videos in the past to complement past columns. I have also done similar videos for my column *What's the Problem?* which I have been writing for the *Ontario Mathematics Gazette* since 2006 (which is very much the parent of *Alternate Angles*). I do a (fairly) regular video series called *Math Shorts*, which is aimed at students from upper-elementary school to high school and presents material that they may not see, but can understand in an entertaining way. The most recent video explores the [geometric mean](#) that we encountered earlier in the article.

In short, if you enjoy this column, there is probably something on *Mathematical Mish Mash* that will appeal to you. Check it out!



Shawn Godin is a retired mathematics teacher and department head living in Carleton Place, Ontario, just outside of Ottawa. He strongly believes in the central role of problem solving in the mathematics classroom. He continues to be involved in mathematical activities: leading workshops, writing articles, working on local projects and helping create mathematics contests. Comments and questions are welcome at GodinMathStuff@gmail.com.